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J. Differential Equations 256 (2014) 2739–2777

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic behavior for a singular diffusion equation with gradient absorption

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Received 22 March 2013

Available online 31 January 2014

Abstract

We study the large time behavior of non-negative solutions to the singular diffusion equation with gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N,$$

for $p_c := 2N/(N+1) < p < 2$ and $p/2 < q < q_* := p - N/(N+1)$. We prove that there exists a unique very singular solution of the equation, which has self-similar form and we show the convergence of general solutions with suitable initial data towards this unique very singular solution.

MSC: 35K67; 35K92; 35B40; 35D40

Keywords: Large time behavior; Singular diffusion; Gradient absorption; Very singular solutions; p-Laplacian; Bounded measures

1. Introduction and results

The aim of the present paper is to study the large time behavior of non-negative solutions to the following equation with singular diffusion and gradient absorption:

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2014.01.016

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad (t,x) \in Q_\infty := (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

for $p_c := 2N/(N+1) < p < 2$ and $p/2 < q < q_* := p - N/(N+1)$. We consider only non-negative initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

under suitable decay and regularity assumptions that will be specified later. Eq. (1.1) presents a competition between the effects of the two terms: one term of singular diffusion $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which in our case is supercritical (that is, $p > p_c = 2N/(N+1)$) in order to avoid extinction in finite time, and another term of nonlinear absorption depending on the gradient $|\nabla u|^q$. Due to this competition, interesting mathematical features appear in some ranges of exponents p and q.

The qualitative theory of (1.1) for general exponents p and q developed very recently; indeed, while there are many (even classical ones) papers on nonlinear diffusion equations with zero order absorption, covering almost all possible cases, the study of the gradient absorption proved to be much more involved and brought a bunch of very interesting mathematical phenomena, some of them having been the subject of intensive research in the last decade. As expected, the first results were obtained in the semilinear case $p = 2$, where the asymptotic behavior for $q > 1$ has been identified in a series of papers [4,5,7,11,12,19,20]. Finite time extinction was shown to take place for $q \in (0,1)$ [8,9,20] while the critical case $q = 1$, in spite of its apparent simplicity, is still far from being fully understood: only some large-time estimates are available [10] but no precise asymptotics. Passing to the p-Laplacian is a natural step, and for the slow-diffusion case $p > 2$, the exponent $q = p - 1$ proved to have a very interesting critical effect, as an interface between absorption-dominated behavior and diffusion-dominated behavior [3,28], while itself gives rise to a critical regularized sandpile-type behavior, as shown recently in [24]. A natural next step was then to pass to the study of the fast-diffusion case $1 < p < 2$, where the authors made important progress recently in understanding the decay rates and typical self-similar profiles [22,23]. In particular, finite time extinction was shown to take place when (p,q) ranges in $(p_c, 2) \times (0, p/2)$ and in $(1, p_c) \times (0, \infty)$ while diffusion is likely to govern the large time dynamics when $(p,q) \in (p_c, 2) \times (q_*, \infty)$. The intermediate range $(p,q) \in (p_c, 2) \times (p/2, q_*)$ features a balance between the diffusion and absorption terms and is the focus of this paper.

From now on, we restrict ourselves to the following range of exponents:

$$p \in (p_c, 2) \quad \text{and} \quad q \in \left(\frac{p}{2}, q_*\right), \tag{1.3}$$

and we set

$$\alpha := \frac{p-q}{2q-p} > 0, \qquad \beta := \frac{q-p+1}{2q-p} > 0 \quad \text{and} \quad \eta := \frac{1}{N(p-2)+p} > 0, \tag{1.4}$$

the positivity of η being a consequence of $p > p_c$. We also observe that, thanks to (1.3),

$$\alpha - N\beta = \frac{(N+1)(q_* - q)}{2q-p} > 0. \tag{1.5}$$

In order to state the main result concerning the large-time behavior, we recall a special category of solutions to (1.1), that are called *very singular solutions*. These are solutions to (1.1)