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Regularity criteria for the three-dimensional magnetohydrodynamic equations

Jishan Fan [a], Fucai Li [b,*], Gen Nakamura [c], Zhong Tan [d]

[a] *Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, PR China*
[b] *Department of Mathematics, Nanjing University, Nanjing 210093, PR China*
[c] *Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea*
[d] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China*

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Abstract

This paper studies the three-dimensional density-dependent incompressible magnetohydrodynamic equations. First, a regularity criterion is proved which allows the initial density to contain vacuum. Then we establish another blow-up criterion in the Besov space $\dot{B}^0_{\infty,2}$ when the positive initial density is bounded away from zero. Third, we prove a global nonexistence result for initial density with high decreasing at infinity. Fourth, we obtain a regularity criterion to the density-dependent incompressible magnetohydrodynamic equations in a bounded domain. Finally, we also give some remarks on the regularity criteria for the three-dimensional full compressible magnetohydrodynamic equations in a bounded domain and for the incompressible homogeneous magnetohydrodynamic equation in the whole space $\mathbb{R}^3$.

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* Corresponding author.
E-mail addresses: fanjishan@njfu.edu.cn (J. Fan), fli@nju.edu.cn (F. Li), gnaka@math.sci.hokudai.ac.jp (G. Nakamura), tan85@xmu.edu.cn (Z. Tan).

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1. Introduction

In this paper we study the regularity criteria for the following density-dependent incompressible magnetohydrodynamic (MHD) equations (see [13,17]):

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) = b \cdot \nabla b, \tag{1.2}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \eta \Delta b, \tag{1.3}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.4}$$

with the initial data

$$(\rho, u, b)(x, 0) = (\rho_0, u_0, b_0)(x), \quad x \in \mathbb{R}^3, \tag{1.5}$$

where the unknowns ρ, u, b and π denote the density of the fluid, the fluid velocity field, the magnetic field and the pressure, respectively. The parameter $\mu > 0$ denotes the viscous coefficient and $\eta \geqslant 0$ the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic field.

The MHD equations (1.1)–(1.4) have been studied by many authors [8,12,1,2,24,7]. When the initial density ρ_0 has a positive bound, Gerbeau and Le Bris [12] and Desjardins and Le Bris [8] obtained the global existence of weak solutions with finite energy in the whole space $\mathbb{R}^3$ and in the torus $\mathbb{T}^3$ respectively. In [1], Abidi and Hmidi obtained the local existence of strong solutions. They also proved the global existence of strong solutions when the initial data are small in some Sobolev spaces. Abidi and Paicu [2] established the local existence of weak solutions in some Besov space and constructed the global solution if the initial data are small. When the initial density ρ_0 may contain vacuum, Wu [24] and Chen et al. [7] obtained the local existence and uniqueness of strong solution to the problem (1.1)–(1.5). More precisely, it is shown that if the initial data ρ_0, u_0, and b_0 satisfy

$$\begin{cases} 0 \leqslant \rho_0 \leqslant M < \infty, \quad \nabla \rho_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), \quad 3 < q \leqslant 6, \\ (u_0, b_0) \in H^2(\mathbb{R}^3), \quad \operatorname{div} u_0 = \operatorname{div} b_0 = 0, \end{cases} \tag{1.6}$$

and if, in addition, the following compatibility condition

$$-\Delta u_0 + \nabla\left(\pi_0 + \frac{1}{2}|b_0|^2\right) - b_0 \cdot \nabla b_0 = \sqrt{\rho_0}\, g \tag{1.7}$$

holds for some function $g \in L^2(\mathbb{R}^3)$, then there exist a positive time $T_* \in (0, \infty]$ and a unique strong solution (ρ, u, b) to the problem (1.1)–(1.5) satisfying

$$\begin{cases} 0 \leqslant \rho \leqslant M, \quad \nabla \rho, \rho_t \in C\big([0, T_*]; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)\big), \\ (u, b) \in C\big([0, T_*]; H^2(\mathbb{R}^3)\big) \cap L^2\big(0, T_*; W^{2,6}(\mathbb{R}^3)\big), \\ \sqrt{\rho} u_t \in L^\infty\big(0, T_*; L^2(\mathbb{R}^3)\big), \quad u_t \in L^2\big(0, T_*; H^1(\mathbb{R}^3)\big), \\ b_t \in L^\infty\big(0, T_*; L^2(\mathbb{R}^3)\big) \cap L^2\big(0, T_*; H^1(\mathbb{R}^3)\big). \end{cases} \tag{1.8}$$