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The Cauchy problem for the generalized Camassa–Holm equation in Besov space

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Abstract

In this paper we consider the Cauchy problem for the generalized Camassa–Holm equation $u_t + u^Q u_x + \partial_x(1-\partial_x^2)^{-1}[2ku + \frac{Q^2+3Q}{2(Q+1)}u^{Q+1} + \frac{Q}{2}u^{Q-1}u_x^2] = 0$ in Besov space. First, we prove that the solutions to the Cauchy problem for the generalized Camassa–Holm equation do not depend uniformly continuously on the initial data in $H^s(\mathbf{R})$ with $s < 3/2$ when $k = 0$. Second, combining the real interpolations among inhomogeneous Besov spaces with Lemma 5.2.1 of [6] which is called Osgood Lemma (a substitute for Gronwall inequality), we show that the Cauchy problem for the generalized Camassa–Holm equation is locally well-posed in $B^{3/2}_{2,1}$. Finally, we give a blow-up criterion.

MSC: 35G25; 35L05; 35R25

Keywords: Cauchy problem; Generalized Camassa–Holm equation; Besov spaces; Osgood Lemma; Blow-up criterion

1. Introduction

In this paper, we are concerned with the study on the Cauchy problem for the generalized Camassa–Holm equation

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$$u_t - u_{txx} + \big(a(u)\big)_x - \left(b'(u)\frac{u_x^2}{2} + b(u)u_{xx}\right)_x = 0, \tag{1.1}$$

where

$$b(u) = u^Q, \qquad a(u) = 2ku + \frac{Q+2}{2}u^{Q+1}, \quad Q \in \mathbf{N}^+. \tag{1.2}$$

(1.1) is introduced by S. Hakkafv and K. Kirchev in [24] and in (1.1). $u(x,t)$ stands for the fluid velocity at time $t \geqslant 0$ in the spatial direction.

When $Q = 1$, (1.1) reduced to the well-known Camassa–Holm equation

$$u_t + 2ku + 3uu_x - u_{xxt} - 2uu_x - uu_{xxx} = 0 \tag{1.3}$$

which describes the motion of shallow water waves and possesses soliton solutions, bi-Hamiltonian and infinitely many conserved integrals [5]. (1.3) was derived by Camassa and Holm [5] and by Fokas and Fuchssteiner [21]. (1.3) has attracted the attention of lots of people, e.g. [1–3,8–19,22,25–29,31–35,37,38]. By using the transport equation theory and the Besov spaces, Danchin [18] proved that the Cauchy problem for the Camassa–Holm equation is locally well-posed in $B^{3/2}_{2,1}$. Recently, Himonas et al. [29] proved that the flow map $u_0 \to u(t)$ for the Camassa–Holm equation is not uniformly continuous from any bounded sets of $H^1(\mathbf{R})$ into $C([-T,T]; H^1(\mathbf{R}))$ by using traveling wave solutions that are smooth except at finitely many points at which the slope is $\pm\infty$ (cuspons). Very recently, Himonas and Kenig [25] proved that the flow map $u_0 \to u(t)$ for the Camassa–Holm equation is not uniformly continuous from any bounded sets of $H^s(\mathbf{R})$ into $C([-T,T]; H^s(\mathbf{R}))$ with $s > 3/2$.

Note that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x) * f = (1-\partial_x^2)^{-1} f$ for all $f \in L^2(\mathbf{R})$ and $G * y = u$, we assume that $P(D) = -\partial_x(1-\partial_x^2)^{-1}$, we can rewrite (1.1) as follows:

$$u_t + u^Q u_x - P(D)\left[2ku + \frac{Q^2+3Q}{2(Q+1)}u^{Q+1} + \frac{Q}{2}u^{Q-1}u_x^2\right] = 0, \quad t > 0. \tag{1.4}$$

We supplement (1.4) with the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{1.5}$$

Hakkafv and Kirchev [24] proved that the Cauchy problem for (1.4) is locally well-posed with data in $H^s(\mathbf{R})$, $s > 3/2$.

In this paper, inspired by [27], when $k = 0$ in (1.1), we prove that the solutions to the Cauchy problem for the generalized Camassa–Holm equation do not depend uniformly continuously on the initial data in $H^s(\mathbf{R})$ with $s < 3/2$. Motivated by [17,19], by virtue of the Littlewood–Paley decomposition and nonhomogeneous Besov spaces and iterative method, we show that the Cauchy problem for (1.4) is locally well-posed in Besov space with data in $B^{3/2}_{2,1}$. The main difficulties that we are encountered with are to prove that the approximate solution in $B^{3/2}_{2,1}$ is uniformly bounded and that the approximate solution is a Cauchy sequence in $B^{1/2}_{2,1}$ due to the fact that the structure of (1.4) is much more complicated than its of Camassa–Holm equation. We