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Uniform persistence and Hopf bifurcations in $\mathbb{R}^n_+$

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Abstract

We consider parameterized families of flows in locally compact metrizable spaces and give a characterization of those parameterized families of flows for which uniform persistence continues. On the other hand, we study the generalized Poincaré–Andronov–Hopf bifurcations of parameterized families of flows at boundary points of $\mathbb{R}^n_+$ or, more generally, of an n-dimensional manifold, and show that this kind of bifurcations produce a whole family of attractors evolving from the bifurcation point and having interesting topological properties. In particular, in some cases the bifurcation transforms a system with extreme non-permanence properties into a uniformly persistent one. We study in the paper when this phenomenon happens and provide an example constructed by combining a Holling-type interaction with a pitchfork bifurcation.

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1. Preliminaries

This paper is devoted to the study of some questions related to persistence of flows. This is a topic classically connected to population dynamics, the central issue being to determine whether some components of the population are over the long term driven to extinction or, on the contrary, they will survive and evolve towards some stable states where coexistence between all the components is achieved. The term persistence is given to systems in which strictly positive solutions do not approach the boundary of the nonnegative orthant $\mathbb{R}^n_+$ as $t \to \infty$. The question that arises is that of determining conditions which prevent solutions from approaching the boundary. As G. Butler, H.I. Freedman and P. Waltman remark in [3], this is of great importance in the modeling of biological populations where such conditions rule out the possibility of one of the populations becoming arbitrarily close to zero in a deterministic model and therefore risking extinction in a more realistic interpretation of the model.

Several forms of persistence have been studied. The so called *uniform persistence* or *permanence* is perhaps the most robust concept, that is, more likely to be maintained under suitable small variations of the system of equations. This is always a desirable consideration from the point of view of applications. However, strictly speaking, uniform persistence is not fully robust, although in [43] it has been proved that some forms of weak robustness always hold. These conditions are expressed using the notion of *continuation*, which is one of the forms that robustness adopts in the context of the Conley index theory [5,6]. Also, in the papers [11,19,23,47], some sufficient conditions for robustness are given. One of the aims of the present paper is to give a characterization of those parameterized families of flows for which uniform persistence continues. Another aim is to study the generalized Poincaré–Andronov–Hopf bifurcations of parameterized families of flows at boundary points of $\mathbb{R}^n_+$ or, more generally, of an n-dimensional manifold. We see that this kind of bifurcations produce a whole family of attractors evolving from the bifurcation point and having interesting topological properties. A possible consequence of the bifurcation is a qualitative change in the persistence properties of the system. In some cases the bifurcation transforms a system with extreme non-permanence properties into a uniformly persistent one. We study in the paper when this phenomenon happens and provide an example constructed by combining a Holling-type interaction with a pitchfork bifurcation.

In the sequel we fix some terminology and state a few results that will be used along the paper. An attractor of a flow $\varphi : E \times \mathbb{R} \to E$, where E is a locally compact metrizable space, is in this paper an asymptotically stable invariant compactum. A repeller is a negatively asymptotically stable invariant compactum, i.e. an attractor for the reverse flow. The following characterization of repellers is useful (see [36]): An invariant compactum K is a repeller if and only if there is a neighborhood U of K in E such that for every $x \in U - K$ there is $t > 0$ such that $\varphi(x,t) \notin U$. This characterization can be dualized for attractors.

The flow φ is said to be dissipative if $\omega(x) \neq \emptyset$ for every $x \in E$ and $\bigcup_{x \in E} \omega(x)$ has compact closure. If E is not compact we shall often consider the Alexandrov compactification $\hat{E} = E \cup \{\infty\}$ and the extended flow

$$\hat{\varphi} : \hat{E} \times \mathbb{R} \to \hat{E}$$

leaving fixed ∞. Then dissipativeness is equivalent to $\{\infty\}$ being a repeller (see [10] and [16]). Notice that the dual attractor of $\{\infty\}$ is a global attractor for the flow φ.

A stronger form of dissipativeness can be given for families of flows. If $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I$, is a (continuous) parameterized family of flows then φ_λ is said to be uniformly dissipative