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Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian

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Abstract

This paper is devoted to the existence of nodal and multiple solutions of nonlinear problems involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ $(n \geqslant 2)$ is a bounded smooth domain, $s \in (0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian. When f is superlinear and subcritical, we prove the existence of a positive solution, a negative solution and a nodal solution. If $f(x,u)$ is odd in u, we obtain an unbounded sequence of nodal solutions. In addition, the number of nodal domains of the nodal solutions are investigated.

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1. Introduction and main results

In this paper we consider nonlinear problems involving the fractional power of the Dirichlet Laplacian

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ $(n \geqslant 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $s \in (0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian. $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Here the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ of a function $\phi \in \mathcal{S}$ is defined by

$$\mathcal{F}\big((-\Delta)^s \phi\big)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi), \quad \forall s \in (0,1),$$

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform, i.e., $\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} \phi(x)\, dx$. If ϕ is smooth enough, it can also be computed by the following singular integral:

$$(-\Delta)^s \phi(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy,$$

where P.V. is the principal value and $c_{N,s}$ is a normalization constant.

The fractional power of the Laplacian $(-\Delta)^s$ is the infinitesimal generator of Lévy stable diffusion processes (see [10]). It arises in several areas such as phase transitions, flames propagation, chemical reaction in liquids, population dynamics, American options in finance, crystal dislocation, one can see [1,22,32] and their references.

Nonlinear equations involving the fractional Laplacian are currently actively studied. Chang and González [19] studied this operator in conformal geometry. Caffarelli et al. [15,16] investigated free boundary problems of the fractional Laplacian. Silvestre [30] obtained some regularity results of the obstacle problem of fractional Laplacian. Since the work of Caffarelli and Silvestre [17], who introduced the s-harmonic extension to define the fractional Laplacian operator, several results of the fractional version of the classical elliptic problems were obtained, one can see [2,11,13,14,20,31] and their references.

On the other hand, in past decades, the existence and multiplicity of positive and nodal solutions of classical elliptic boundary value problems have been widely investigated, one can see [4,8,9,21,24,25,27,33]. Specially, some results on nodal solutions of nonlinear elliptic equations involving different operators have been obtained by combing minimax method with invariant sets of descending flow, such as Laplacian operator [3,26], p-Laplacian operator [5,7] and Schrödinger operator [6,28]. However, due to the fact that the fractional Laplacian operator is nonlocal, very few things on this topic are known about the fractional Laplacian. In this paper, we generalize the method of invariant sets of descending flow to the situation of fractional Laplacian and study the nodal and multiple solutions for the elliptic equations involving fractional powers of the Laplacian $(-\Delta)^s$ for all $s \in (0,1)$. Since the operator $(-\Delta)^s$ is nonlocal, we shall study the nodal solutions by applying the s-harmonic extension to transform the nonlocal problem in Ω to a local problem in the half cylinder $\Omega \times (0,+\infty)$. To construct invariant sets