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Elliptic and parabolic equations in fractured media

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Abstract

The elliptic and the parabolic equations with Dirichlet boundary conditions in fractured media are considered. The fractured media consist of a periodic connected high permeability sub-region and a periodic disconnected matrix block subset with low permeability. Let $\epsilon \in (0, 1]$ denote the size ratio of the matrix blocks to the whole domain and let $\omega^2 \in (0, 1]$ denote the permeability ratio of the disconnected subset to the connected sub-region. It is proved that the $W^{1,p}$ norm of the elliptic and the parabolic solutions in the high permeability sub-region are bounded uniformly in ω, ϵ. However, the $W^{1,p}$ norm of the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. For the elliptic and the parabolic equations in periodic perforated domains, it is also shown that the $W^{1,p}$ norm of their solutions are bounded uniformly in ϵ.

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1. Introduction

The $W^{1,p}$ estimates for the solutions of the elliptic and the parabolic equations with Dirichlet boundary conditions in fractured media are concerned. The problem arises from two-phase problems, flows in fractured media, and the stress in composite materials (see [3,9,15]). Let Ω be a smooth simply-connected domain in $\mathbb{R}^n$ for $n \geq 3$, $\partial\Omega$ be the boundary of Ω, $Y \equiv (0, 1)^n$ consist of a smooth sub-domain Y_m completely surrounded by another connected sub-domain

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Y_f $(\equiv Y \setminus \overline{Y_m})$, $\epsilon \in (0,1]$, $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) \geq 2\epsilon\}$, $\Omega_m^\epsilon \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ be a disconnected subset of Ω, Ω_f^ϵ $(\equiv \Omega \setminus \overline{\Omega_m^\epsilon})$ denote a connected sub-region of Ω, and $\mathbf{K}_{\nu,\epsilon}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\epsilon \\ \nu & \text{if } x \in \Omega_m^\epsilon \end{cases}$ for any $\nu, \epsilon > 0$.

The elliptic equation that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U + G) = F & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0,1]$ and G, F are given functions. If G, F are bounded, a solution of (1.1) in Hilbert space $H^1(\Omega)$ exists uniquely for each ω, ϵ by Lax–Milgram Theorem [12]. The L^2 norm of the gradient of the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in ω, ϵ if G, F are small in Ω_m^ϵ. However, the L^2 norm of the gradient of the solution of (1.1) in matrix blocks Ω_m^ϵ can be very large when ω closes to 0. The parabolic equation that we consider is, for any $\omega, \epsilon \in (0,1]$,

$$\begin{cases} \partial_t U - \nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U) = F & \text{in } \Omega \times (0,T), \\ U = 0 & \text{on } \partial\Omega \times (0,T), \\ U(x,0) = U_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

If F, U_0 are smooth, a solution of (1.2) in Hilbert space $L^2([0,T]; H^1(\Omega))$ exists uniquely for each ω, ϵ. The L^2 norm of the gradient of the solution of (1.2) in the connected sub-region $\Omega_f^\epsilon \times (0,T)$ is bounded uniformly in ω, ϵ if F is small in $\Omega_m^\epsilon \times (0,T)$. However, the L^2 norm of the gradient of the solution of (1.2) in matrix blocks $\Omega_m^\epsilon \times (0,T)$ can be very large when ω closes to 0. One also notes that for the elliptic and the parabolic equations in periodic perforated domains, the H^1 norm of their solutions are bounded uniformly in ϵ.

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [15,18]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates for uniform elliptic equations with Hölder periodic coefficients were shown in [4,5]. Uniform $W^{1,p}$ estimate for uniform elliptic equations with continuous periodic coefficients was considered in [6] and the same problem with VMO periodic coefficients could be found in [22]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [19] and the same problem in Lipschitz estimate was studied in [21]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform parabolic equations with oscillating periodic coefficients were obtained in [10]. For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [13]. Uniform Hölder estimate in ϵ for non-uniform parabolic equations with discontinuous periodic coefficients was shown in [23].

Here we present uniform $W^{1,p}$ estimate for the solutions of the non-uniform elliptic and the non-uniform parabolic equations with Dirichlet boundary conditions in fractured media. It is proved that the $W^{1,p}$ norm of the elliptic and the parabolic solutions in the high permeability sub-region Ω_f^ϵ are bounded uniformly in ω, ϵ. However, the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. For the elliptic and the parabolic equations in perforated domains, it is also shown that the $W^{1,p}$ norm of their solutions are bounded uniformly in ϵ. A three-step compactness argument introduced in [4,5] will be employed to obtain the