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On uniform and decay estimates for unbounded solutions of partial differential equations

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Abstract

We prove that if a function u satisfies certain integral estimates (of energy type) then it verifies estimates of the type

$$\|u(t)\|_{L^r(\Omega)} \leq c\frac{\|u(0)\|^{\gamma_0}_{L^{r_0}(\Omega)}}{t^{\gamma_1}}, \qquad t > 0,$$

where r and r_0 are exponents that appear in these integral estimates and γ_1 and γ_0 are positive constants that can be expressed in terms of these constants. We will see that in some cases $r > r_0$ (supercontractive estimates) and hence a regularizing effect on u appears while in other cases this improvement of regularity does not appear since $r < r_0$; in any case, we prove that the $L^r(\Omega)$-norm of u decays in time (for t large) since it verifies the previous estimate.

We show how to apply this result to obtain new estimates of this kind for the solutions of many nonlinear parabolic equations. This new method allows also to explain the reason of the similar behavior of the solutions of very different parabolic problems.

Finally, we study sufficient conditions for extinction in finite time.

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1. Introduction

It is well known that the solution u of the heat equation

$$\begin{cases} u_t = \triangle u & \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^N$ ($N \geq 3$) and u_0 is only a summable function, becomes immediately bounded and satisfies the following decay (or ultracontractive) estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}} \quad \text{for every } t > 0.$$

The same "strong regularizing" effect, that is, using the words of Brezis and Crandall: "for every $t > 0$ we have $u(t) \in L^\infty(\Omega)$ if only $u_0 \in L^1(\Omega)$" (see [13]) appears for many other parabolic problems, linear or nonlinear, degenerate or singular and even doubly nonlinear (see [32] and the references therein). Many mathematicians have studied this strong regularizing effect and numerous interesting results can be found in literature for many parabolic problems like, for example, the p-Laplacian equation (see [41,25,18,19,26,40] and the references therein), the porous medium equation (see [5,4,41,6,1,40] and the references given there) the fast diffusion equation [5,41,24], the doubly nonlinear equation (degenerate and singular) (see [16,11,32] and the references therein), parabolic problems of Leray–Lions type (see [15,32,30]) and also other evolution equations (see for example [12,20,34] and the references therein).

In [32] we have proved that this strong regularization appears when the solutions satisfy suitable integral estimates. This result explains why different equations have similar behavior and allows to derive these estimates, with a unified and easy proof, for all the cited well known problems together with other interesting equations.

We recall that the L^∞-regularity is the first step in all further regularity properties, like Hoelder continuity, etc. Moreover, it can be useful in proving uniqueness results (see [13,30]).

Aim of this paper is to study what happens when this strong regularizing effect does not appear. In detail, we will focus our attention in which kind of regularity have the solutions and if they can decay in some Lebesgue space (different from L^∞).

We point out that decay estimates for unbounded solutions are less known in literature (see for example, [21,10,22,23] and the references therein).

We prove here that many different behaviors can appear.

For example, it can happen that the solution does not regularize at all since it remains in the space where is the initial datum u_0, i.e.

$$u_0 \in L^{r_0}(\Omega) \quad \Longrightarrow \quad u(t) \in L^{r_0}(\Omega) \quad \text{and} \quad u(t) \notin L^r(\Omega) \quad \text{for every } r > r_0.$$

Examples of this "no-regularization" are some solutions of the singular p-Laplacian equation

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.2}$$