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Limit cycles bifurcating from planar polynomial quasi-homogeneous centers

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Abstract

In this paper we find an upper bound for the maximum number of limit cycles bifurcating from the periodic orbits of any planar polynomial quasi-homogeneous center, which can be obtained using first order averaging method. This result improves the upper bounds given in [7].

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1. Introduction

In this work we deal with polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$. The dot denotes derivative with respect to an independent real variable t. We say that the degree of the system is $n = \max\{\deg P, \deg Q\}$.

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Let $\mathbb{N}$ denote the set of positive integers. The polynomial differential system (1) is *quasi-homogeneous* if there exist $p, q, m \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}$,

$$P(\alpha^p x, \alpha^q y) = \alpha^{p+m-1} P(x, y), \;\; Q(\alpha^p x, \alpha^q y) = \alpha^{q+m-1} Q(x, y), \tag{2}$$

where p and q are called the *weight exponents* of system (1), and m is the *weight degree* with respect to the weight exponents p and q. We say that system (1) satisfying conditions (2) is a quasi-homogeneous system of weight (p, q, m). We remark that for the particular case $p = q = 1$, system (1) is the classical homogeneous polynomial differential system of degree m. We note that conditions (2) imply that the origin of coordinates is a singular point of system (1). We will first characterize when the origin of system (1) is a center (that is, it has a neighborhood filled with periodic orbits with the exception of the origin). Lemma 4 statement (ii) (see below) characterizes all the centers of the quasi-homogeneous systems. This characterization is well-known see for instance [7]. Moreover in [1] the authors provide an algorithm for obtaining the quasi-homogeneous systems with a given degree which is a combinatorial problem.

When the origin of system (1) is a center, we consider the one-parametric family of systems

$$\dot{x} = P(x, y) + \varepsilon \bar{P}(x, y), \quad \dot{y} = Q(x, y) + \varepsilon \bar{Q}(x, y), \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is the perturbation parameter and $\bar{P}$ and $\bar{Q} \in \mathbb{R}$ are arbitrary polynomials of degree n. Our goal is to give the maximum number of limit cycles which can bifurcate from the periodic orbits of the center localized at the origin of system (1) with $\varepsilon = 0$, inside the family (3) for $\varepsilon \neq 0$ sufficiently small. We can give this maximum number in terms of p, q and n.

Moreover our result is a generalization of the one given in Theorem A of [7] because we do not need the hypothesis that the polynomials P and Q are coprime. Indeed, we will use the averaging method at first order of ε instead of the Abelian integral which is used in [7]. In our approach we use the classical trigonometric functions $\sin\theta$ and $\cos\theta$ instead of the generalized trigonometric functions $Cs\,\theta$ and $Sn\,\theta$ which are related to quasi-homogeneous functions. In our work the integral that we find, see (5), instead of the Abelian integral is an integral of elementary functions.

Our main result is the following one.

Theorem 1. *We consider any quasi-homogeneous polynomial differential system of weight (p, q, m) of the form (1) having a center at the origin. We denote by $(p^*, q^*) = (p/M, q/M)$ where $M = \gcd(p, q)$. We perturb system (1) inside the class of all polynomial differential systems of degree n, that is, we consider systems of the form (3) where $\bar{P}$ and $\bar{Q}$ are arbitrary polynomials of degree n. We assume that $n \geq p^* \geq q^* \geq 1$. Then, the maximum number of zeros $r_0 > 0$ taking into account their multiplicity of the first averaging function* (see (5)) *is at most*

(a) $(2(n+1)p^* - p^{*2} + (3 + (-1)^{n+1})p^* - 7)/4$ *if p^* and q^* are odd;*
(b) $(2(n+1)p^* - p^{*2} + 2p^* - 4)/4$ *if p^* is even and q^* is odd; and*
(c) $(2(n+1)p^* - p^{*2} + (-1)^n + 2p^* - 4)/4$ *if p^* is odd and q^* is even.*

Statements (b) and (c) of Theorem 1 are improvements of the ones given in [7] because the expression $2p^* - 4$ there appears as $4p^* - 8$.

The upper bounds provided in Theorem 1 cannot always be reached, as the following result shows.