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Pulsating waves of a partially degenerate reaction–diffusion system in a periodic habitat ☆

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Abstract

We investigate a partially degenerate reaction–diffusion system in a periodic habitat and prove the existence and stability of pulsating waves. More specifically, we show that if the wave speed is greater than the spreading speed, then there exists a pulsating wave connecting the stable positive periodic steady state to the unstable trivial one. Further, this pulsating wave attracts exponentially in time all solutions with initial functions in its bounded neighborhood with respect to a weighted maximum norm.

MSC: 35K57; 92D25

Keywords: Pulsating waves; Partially degenerate; Reaction–diffusion system; Periodic habitat

1. Introduction

Pulsating waves (also called spatially periodic traveling waves in the literature) were first introduced in [11] to study the invasion of a new migrating species in a heterogeneous environment. The definitions of pulsating waves vary but are equivalent in previous studies (see, e.g., [3,10,18]). Here we adopt the following generalized definition.

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Definition 1.1. For a reaction–diffusion system of general type:

$$\partial_t u = \mathcal{A}(x)u + \mathcal{F}(x,u),\ t \geq 0,\ \ x \in \mathbb{R}^n, u = (u_1, \cdots, u_m), \tag{1.1}$$

where $\mathcal{A} = (\mathcal{A}_1, \cdots, \mathcal{A}_m)$ is a linear diffusion operator (involving local differential operators or/and nonlocal integral operators) and $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_m)$ is a nonlinear operator, assume that $\mathcal{A}$ and $\mathcal{F}$ are periodic in x with the same period. A solution $u(t,x)$ is called a *pulsating wave* connecting two periodic steady states $p_-(x)$ and $p_+(x)$ with speed c and direction e provided that

(i) $u(t,x)$ has the special form $u(t,x) = \Psi(x \cdot e - ct, x)$, where e is a unit vector in $\mathbb{R}^n$ and $\Psi(s,x)$ is periodic in the second variable x with the same period as $\mathcal{A}$ and $\mathcal{F}$.
(ii) As $s \to \pm\infty$, $\Psi(s,x)$ tends to $p_\pm(x)$, respectively, uniformly in $x \in \mathbb{R}^n$.

Note that when $m = 1$ (i.e., the scalar case) and the diffusion term $\mathcal{A}(x)u$ has the divergent form $\nabla \cdot (A(x)\nabla u)$, system (1.1) was investigated by Berestycki, Hamel and Roques [1,2], where they proved the existence and uniqueness of stationary solution and analyzed asymptotic behavior of solutions. Furthermore, they obtained the existence of pulsating waves and a variational characterization of the minimal wave speed.

A nonlocal and time-delayed population model in a periodic habitat was proposed by Weng and Zhao [16], and the authors studied the spatial dynamics of the model system, the global attractiveness of spatially periodic steady state, and the existence of spreading speeds and pulsating waves. This work was further extended by Ouyang and Ou [8] to obtain the stability and convergence rate of pulsating waves.

For the general case ($m \geq 1$), if the solution maps associated with (1.1) are compact with respect to the compact open topology, or more weakly, if the solution maps are α-contractions with respect to the Kuratowski measure of non-compactness, then one may use the abstract theory developed by Weinberger [14,15] and Liang and Zhao [5,6] to show that a pulsating wave exists if and only if the wave speed c is no less than the spreading speed. A natural question is whether this result (i.e., the spreading speed coincides with the minimal wave speed) can be extended to the non-compact case. For the special case of scalar equations ($m = 1$) with a nonlocal integral diffusion operator, the authors of [3,10] gave an affirmative answer. Our purpose is to study a class of partially-degenerate systems, namely, $m > 1$ and some components of the diffusion operator $\mathcal{A} = (\mathcal{A}_1, \cdots, \mathcal{A}_m)$ vanish. For simplicity, we consider the following partially degenerate reaction–diffusion system in a periodic and one-dimensional habitat:

$$\begin{aligned} &\partial_t u_1(t,x) = D_1(x)\partial_{xx}u_1 + D_2(x)\partial_x u_1 + f(x, u_1, u_2), \\ &\partial_t u_2(t,x) = g(x, u_1, u_2). \end{aligned} \tag{1.2}$$

This system is motivated by the benthic–pelagic population model proposed by Lutscher, Lewis and McCauley [7], where u_1 and u_2 are the densities of individuals in the pelagic and benthic zones, respectively. Note that the benthic individuals u_2 do not have any diffusion term, which leads to the non-compactness of solution maps. So we may not expect to apply the abstract results in [5,6,15] to study the spatial dynamics of system (1.2). Recently, Wu, Xiao and Zhao [17] established the existence of spreading speeds by combining the theory of monotone dynamical systems and the ideas in [9]. As a consequence, we can easily conclude that (1.2) admits no