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On the existence of low regularity solutions to semilinear generalized Tricomi equations in mixed type domains ☆

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Abstract

In [20,21], we have established the existence and singularity structures of low regularity solutions to the semilinear generalized Tricomi equations in the degenerate hyperbolic regions and to the higher order degenerate hyperbolic equations, respectively. In the present paper, we shall be concerned with the low regularity solution problem for the semilinear mixed type equation $\partial_t^2 u - t^{2l-1}\Delta u = f(t, x, u)$ with an initial data $u(0, x) = \varphi(x) \in H^s(\mathbb{R}^n)$ $(0 \le s < \frac{n}{2})$, where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \ge 2$, $l \in \mathbb{N}$, $f(t, x, u)$ is C^1 smooth in its arguments and has compact support with respect to the variable x. Under the assumption of the subcritical growth of $f(t, x, u)$ on u, we will show the existence and regularity of the considered solution in the mixed type domain $[-T_0, T_0] \times \mathbb{R}^n$ for some fixed constant $T_0 > 0$.

MSC: 35L70; 35L65; 35L67; 76N15

Keywords: Generalized Tricomi equation; Mixed type equation; Confluent hypergeometric function; Modified Bessel functions; Calderón–Zygmund decomposition; Multiplier

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1. Introduction

In [20,21], we have established the existence and singularity structures of low regularity solutions to the semilinear generalized Tricomi equations in the hyperbolic regions and to the higher order degenerate hyperbolic equations, respectively. In the present paper, we have a further study on the existence and regularities of solutions to the following n-dimensional semilinear generalized Tricomi equation in the mixed type domain $\mathbb{R} \times \mathbb{R}^n$

$$\begin{cases} \partial_t^2 u - t^{2l-1}\Delta u = f(t,x,u), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $l \in \mathbb{N}$, $x = (x_1, \dots, x_n)$, $n \geq 2$, $\Delta = \sum_{i=1}^{n} \partial_i^2$, $\varphi(x) \in H^s(\mathbb{R}^n)$ $(0 \leq s < \frac{n}{2})$, $f(t,x,u)$ is C^1 smooth in its arguments and has a compact support E on the variable x. Moreover, for any $T > 0$, there exists $C_T > 0$ such that for $(t,x,u) \in [-T,T] \times E \times \mathbb{R}$,

$$|f(t,x,u)| \leq C_T(1+|u|)^{\mu} \quad \text{and} \quad |\partial_u f(t,x,u)| \leq C_T(1+|u|)^{\max\{\mu-1,0\}}, \tag{1.2}$$

where $C_T > 0$ is a constant depending only on T, and the fixed constant exponent $\mu \geq 0$ fulfills

$$\mu < p_0 \equiv \frac{2n}{n-2s}. \tag{1.3}$$

Here we point out that the number p_0 defined in (1.3) comes from the Sobolev embedding formula $H^s(\mathbb{R}^n) \subset L^{p_0}(\mathbb{R}^n)$. Thus (1.2) and (1.3) mean that the nonlinearity f in (1.1) admits a "subcritical" growth on the variable u. In addition, we shall illustrate that the scope of the exponent μ for solving problem (1.1) is closely related to the number $Q_0 \equiv 1 + \dfrac{n(2l+1)}{2}$. In the terminology of [17] and the references therein, Q_0 is called the homogeneous dimension corresponding to the degenerate elliptic operator $\partial_t^2 - t^{2l-1}\Delta$ for $t \leq 0$. Our main result in this paper is:

Theorem 1.1. *Under the assumptions of (1.2)–(1.3) and $\frac{2}{2l+1} \leq s < \min\{\frac{n}{2}, \frac{4}{2l+1}\}$, there exists a constant $T_0 > 0$ such that problem (1.1) has a solution $u \in C([-T_0,0], L^{p_0}(\mathbb{R}^n)) \cap C([0,T_0], L^{p_1}(\mathbb{R}^n))$ when $0 \leq \mu \leq 1$, or when $1 < \mu < p_0$ and $Q_0 \leq \frac{p_1}{\mu-1}$, where $\frac{1}{p_1} = \frac{1}{2} - \frac{1}{n}(s - \frac{2}{2l+1})$.*

Remark 1.1. The restriction $Q_0 \leq \frac{p_1}{\mu-1}$ seems necessary when $1 < \mu < p_0$ is posed in Theorem 1.1, otherwise, the standard iteration scheme for solving problem (1.1) only works in finite steps or the solution $u \notin C([-T_0,0], L^{p_0}(\mathbb{R}^n))$ when $Q_0 > \frac{p_0}{\mu-1}$. One can see Remark 4.1 and the related explanations (4.7) in Section 4, respectively.

Remark 1.2. For $l = 1$, (1.1) is the well-known semilinear Tricomi equation $\partial_t^2 u - t\Delta u = f(t,x,u)$. When an initial data $u(0,x) = \varphi(x) \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ is given and the crucial assumption of *supp* $f \subset \{t \geq 0\}$ is posed (namely, $f \equiv 0$ holds in $t \leq 0$, which means that the related Tricomi equation is linear in the elliptic region $\{t \leq 0\}$), M. Beals in [2] shows that