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Moore–Gibson–Thompson equation with memory, part II: General decay of energy

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Abstract

We study a temporally third order (Moore–Gibson–Thompson) equation with a memory term. Previously it was known that, in non-critical regime, the global solutions exist and the energy functionals decay to zero. More precisely, it is known that the energy has exponential decay if the memory kernel decays exponentially. The current work is a generalization of the previous one (Part I) in that it allows the memory kernel to be more general and shows that the energy decays the same way as the memory kernel does, exponentially or not.

Keywords: Moore–Gibson–Thompson equation; Memory damping; Energy estimate; Convex function; Decay rate

1. Introduction

We study the energy decay of Moore–Gibson–Thompson (MGT) equation with a viscoelastic term

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b\mathcal{A} u_t - \int_0^t g(t-s)\mathcal{A} u(s)ds = 0, \tag{1}$$

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with initial data

$$u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2, \tag{2}$$

where τ, c, b are parameters inherited from modeling process, see [1] and references therein. The constant α can be scaled out; we keep it, however, for notational consistency with [1]. $\mathcal{A}$ is a positive self-adjoint operator defined in a real Hilbert space H. The convolution term $\int_0^t g(t-s)\mathcal{A}u(s)ds$ reflects the memory effect of viscoelastic materials; the "memory kernel" $g(t) : [0, \infty) \to [0, \infty)$ directly relates to whether or how the energy decays. Without this memory term, it is known the MGT equation has exponential energy decay in the non-critical regime, where $\gamma = \alpha - \frac{c^2\tau}{b} > 0$, see [1].

In an earlier work [2], we studied (1) with a nontrivial $g(t)$, but focusing on the case where $g(t)$ has exponential decay. We were able to get exponential decay of the energy for three types of memories in the non-critical regime. Here we study the case where the memory kernel has a more general decay rate. For the sake of clarity, in this work we restrict our attention to one of the three types of memories introduced in [2].

Notations:
- $g(t)$: memory kernel.
- $G(t) = \int_0^t g(s)ds$: strength of memory.
- H: real Hilbert space.
- $||\cdot||$: norms on H.
- $g \circ h \triangleq \int_0^t g(t-s)||h(t) - h(s)||^2 ds$, $g(\cdot) \in C(R^+)$, $h(\cdot) \in H$.

1.1. Main results

Our work shows that the energy decay rate of system (1), where memory effects get involved, is determined solely by the memory kernel: if $g(t)$ decays exponentially, then the energy decays exponentially too; if $g(t)$ decays slower, then the energy decays slower as well.

Assumption 1.1. *Let $G(t) = \int_0^t g(s)ds$. We assume*

1. $g(t) \in C^1(\mathbb{R}_+)$, $g(t) > 0$, $g(0) < \frac{b\alpha\gamma}{\tau^2}$ *and* $G(+\infty) < c^2$.
2. *There exists a convex function $H(\cdot) \in C^1(\mathbb{R}_+)$, which is strictly increasing with $H(0) = 0$, such that*

$$g'(t) + H(g(t)) \le 0, \forall t > 0.$$

3. *Let $y(t)$ be a solution of the following ODE*

$$y'(t) + H(y(t)) = 0, y(0) = g(0),$$

and there exists $\alpha_0 \in (0, 1)$ such that $y^{1-\alpha_0}(\cdot) \in L_1(\mathbb{R}_+)$.

4. *There exists $\bar{\delta} > 0$ such that $H(\cdot) \in C^2(0, \bar{\delta})$ and $x^2H''(x) - xH'(x) + H(x) \ge 0$, $\forall x \in [0, \bar{\delta}]$.*
5. *$\mathcal{A}$ satisfies $||u|| \le \lambda_0||\mathcal{A}^{1/2}u||$ for all $u \in H$.*
6. $\gamma = \alpha - \frac{c^2\tau}{b} > 0$.