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Integral representations of a class of harmonic functions in the half space

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Abstract

In this article, motivated by the classic Hadamard factorization theorem about an entire function of finite order in the complex plane, we firstly prove that a harmonic function whose positive part satisfies some growth conditions, can be represented by its integral on the boundary of the half space. By using Nevanlinna's representation of harmonic functions and the modified Poisson kernel of the half space, we further prove a representation formula through integration against a certain measure on the boundary hyperplane for harmonic functions not necessarily continuous on the boundary hyperplane whose positive parts satisfy weaker growing conditions than the first question. The result is further generalized by involving a parameter m dealing with the singularity at the infinity.

Keywords: Integral representation; Positive part; Modified Poisson kernel

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1. Introduction

Some fundamental properties of entire functions of finite order and type in the complex plane or analytic functions in the right (upper) half-plane, have been well studied (see [2,10]). In light of results from Complex Analysis, the order of a classic harmonic function with the Poisson integral in the half space of $\mathbb{R}^n$ is 1, if we define the order of harmonic functions in higher-dimensions similarly to that of entire functions. In what follows, $\mathbb{H} = \{x \in \mathbb{R}^n : x = (x', x_n), x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ represents the upper half space of $\mathbb{R}^n$.

However, when the order is greater than 1, as far as we know, there has been one paper concerning this higher-dimensional problem: The recent paper [15] establishes an integral representation for harmonic functions in $\mathbb{H}$ with order less than 2, by using Carleman's formula and Nevanlinna's representation [16]. The latter mentioned two formulas in one complex variable were useful in the classical theory of functions of one complex variable. Paper [16] generalized the Carleman's formula for harmonic functions in the half plane to the higher-dimensional half space, and established a Nevanlinna's representation for harmonic functions in the half sphere by using Hörmander's theorem, so they are invaluable tools in the study of harmonic functions in the half space $\mathbb{H}$ as well.

The classic Hadamard factorization theorem of an entire function of finite order [3] and the inner and outer factorization theorem of analytic functions in the Hardy spaces in a half plane [6,9] motivate us to carry out this study on harmonic functions in higher-dimensional spaces as given in the forthcoming two sections. Such a higher-dimensional situation is important, interesting and worthwhile for further investigation. In Section 2 we employ Carleman's formula [16] to give the integral representation of harmonic functions with order less than 3, where integral boundary conditions are assumed in place of growth conditions describing the finite order or type properties for entire functions. We also prove that a harmonic function with a finite order, not necessarily continuous on the boundary hyperplane, has an integral representation involving a measure. We make use of Nevanlinna's representation [16] and the modified Poisson kernel of the half space $\mathbb{H}$ [5]. Integral boundary conditions are used to displace the terminology of finite order as well. In Section 3 we provide proofs of the main results.

2. Preliminaries

The notation and terminology that are used in this article can be found in [4,15].

Recall that $\mathbb{H}$ is the Euclidean half space, we then have the hyperplane $\mathbb{R}^n = \{x \in \mathbb{R}^n : x = (x', x_n), x_n = 0\}$, which will be denoted as $\partial\mathbb{H}$. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let θ be the angle between x and $\hat{e}_n$, i.e., $x_n = |x|\cos\theta$, $|x'| = |x|\sin\theta$ $(0 \le \theta < \frac{\pi}{2})$, $x \in \mathbb{H}$. We will write $x = x_1\hat{e}_1 + \cdots + x_{n-1}\hat{e}_{n-1} + x_n\hat{e}_n$, where $\hat{e}_i$ is the ith unit coordinate vector and $\hat{e}_n$ is the normal to $\partial\mathbb{H}$.

For a measurable function u on $\partial\mathbb{H}$, the Poisson integral

$$P[u](x) = \frac{2x_n}{n\omega_n} \int_{\partial\mathbb{H}} \frac{u(y')}{|x - y'|^n} dy' \tag{2.1}$$

will exist and then define a harmonic function in $\mathbb{H}$ if [1,12]