Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 971–990
Journal of Differential Equations
www.elsevier.com/locate/jde

Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with cubic polynomials [☆]

Yun Tian [a,b], Pei Yu [a,*]

[a] Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7 Canada
[b] Department of Mathematics, Shanghai Normal University, Shanghai, 200234, PR China

Received 1 November 2013; revised 4 September 2015

Available online 19 September 2015

Abstract

This paper contains two parts. In the first part, we shall study the Abelian integrals for Żołądek's example [13], in which the existence of 11 small-amplitude limit cycles around a singular point in a particular cubic vector field is claimed. We will show that the bases chosen in the proof of [13] are not independent, which leads to failure in drawing the conclusion of the existence of 11 limit cycles in this example. In the second part, we present a good combination of Melnikov function method and focus value (or normal form) computation method to study bifurcation of limit cycles. An example by perturbing a quadratic Hamiltonian system with cubic polynomials is presented to demonstrate the advantages of both methods, and 10 small-amplitude limit cycles bifurcating from a center are obtained by using up to 5th-order Melnikov functions.

MSC: 34C07; 34C23

Keywords: Bifurcation of limit cycles; Żołądek's example; Higher-order Melnikov function; Hamiltonian system; Focus value

☆ The first draft of this paper has been posted on arXiv.org since November 15, 2013, No. 1311.3381v2.

* Corresponding author.
E-mail addresses: ytian56@uwo.ca (Y. Tian), pyu@uwo.ca (P. Yu).

http://dx.doi.org/10.1016/j.jde.2015.09.016

1. Introduction

The well-known Hilbert's 16th problem [1] has been studied for more than one century, and the research on this problem is still very active today. To be more specific, consider the following planar system:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials in x and y. The second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that system (1) can have.

The progress in the solution of the problem is very slow. Even the simplest case $n = 2$ has not been completely solved, though in the early 1990's, Ilyashenko [2] and Écalle [3] independently proved that the number of limit cycles is finite for any given planar polynomial vector field. For general quadratic polynomial systems, the best result is $H(2) \geq 4$, obtained more than 30 years ago [4,5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7,8]. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert's 16th problem, the so-called weakened Hilbert's 16th problem was posed by Arnold [10]. The problem is to ask for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h, \delta) = \oint\limits_{H(x,y)=h} Q(x, y)\, dx - P(x, y)\, dy, \tag{2}$$

where $H(x, y)$, $P(x, y)$ and $Q(x, y)$ are all real polynomials in x and y, and the level curves $H(x, y) = h$ represent at least a family of closed orbits for $h \in (h_1, h_2)$, and δ denotes the parameters (or coefficients) involved in P and Q. The weakened Hilbert's 16th problem itself is a very important and interesting problem, closely related to the study of limit cycles in the following near-Hamiltonian system [11]:

$$\dot{x} = H_y(x, y) + \varepsilon\, P(x,\ y), \qquad \dot{y} = -H_x(x, y) + \varepsilon\, Q(x,\ y), \tag{3}$$

where $0 < \varepsilon \ll 1$. Studying the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the Melnikov function $M(h, \delta)$, given in (2).

When we focus on the maximum number of small-amplitude limit cycles, denoted by $M(n)$, bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was proved by Bautin in 1952 [12]. For $n = 3$, several results have been obtained (e.g. see [13–15]). Among them, in 1995 Żołądek [13] first constructed a rational Darboux integral to study existence of 11 small-amplitude limit cycles in cubic vector fields. This pioneer work later motivated many researches in this area to study bifurcation of limit cycles. The rational Darboux integral in [13] is given by