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Box products in nilpotent normal form theory: The factoring method

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Abstract

Let N be a nilpotent matrix and consider vector fields $\dot{\mathbf{x}} = N\mathbf{x} + \mathbf{v}(\mathbf{x})$ in normal form. Then $\mathbf{v}$ is equivariant under the flow e^{N^*t} for the inner product normal form or e^{Mt} for the $\mathfrak{sl}_2$ normal form. These vector equivariants can be found by finding the scalar invariants for the Jordan blocks in N^* or M; taking the *box product* of these to obtain the invariants for N^* or M itself; and then *boosting* the invariants to equivariants by another box product. These methods, developed by Murdock and Sanders in 2007, are here given a self-contained exposition with new foundations and new algorithms yielding improved (simpler) Stanley decompositions for the invariants and equivariants. Ideas used include transvectants (from classical invariant theory), Stanley decompositions (from commutative algebra), and integer cones (from integer programming). This approach can be extended to covariants of $\mathfrak{sl}_2^k$ for $k > 1$, known as SLOCC in quantum computing.

Keywords: Normal forms; Vector fields; Nilpotent linear part; Stanley decompositions; Invariant theory; Stanley decompositions

1. Introduction

Let N be an $n \times n$ real nilpotent matrix, and consider smooth systems of differential equations in $\mathbb{R}^n$ of the form

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$$\dot{\mathbf{x}} = N\mathbf{x} + \mathbf{v}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{v}$ is **strictly nonlinear** ($\mathbf{v}(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^n$ and $\mathbf{v}'(\mathbf{0}) = 0 \in \mathfrak{gl}_n$). We usually regard (1.1) as a **formal system**, that is, $\mathbf{v}(\mathbf{x})$ is expanded in a formal power series (which will contain only terms of degree ≥ 2). Choose a **normal form style** for such systems, either the $\mathfrak{sl}_2$ style, the inner product style, or the simplified style (see Section 2). For a given N and a given normal form style, the **weak description problem** calls for a description of the set of all $\mathbf{v}$ such that (1.1) is in normal form; the **strong description problem** calls for a procedure that generates all such systems exactly once, without repetition. For instance, if $N = N_2$ is the 2×2 nilpotent matrix in upper Jordan form, the solution of both description problems for the inner product normal form style can be given by the expression

$$\mathbf{v} \in \mathbb{R}[[x_1]]x_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \mathbb{R}[[x_1]]x_1^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{1.2}$$

called a **Stanley decomposition**. The meaning of (1.2) is that:

1. A system (1.1) with $N = N_2$ is in inner product normal form if and only if it can be written as

$$\mathbf{v}(\mathbf{x}) = f(x_1)x_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g(x_1)x_1^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{1.3}$$

   where f and g are formal power series in x_1. Therefore (1.2) solves the weak description problem.
2. No two distinct systems of the form (1.3) will be equal (as formal power series). Thus all systems in normal form are generated exactly once as f and g range over $\mathbb{R}[[x_1]]$, and (1.2) also solves the strong description problem.

1.1. Remark. For the more familiar semisimple case, where N is replaced by a matrix S that is diagonalizable over $\mathbb{C}$, the weak description problem is solved by requiring that $\mathbf{v}$ contain only resonant vector monomials (in appropriate complex variables). This does not solve the strong description problem, since there are relations among the resonant monomials allowing the same vector field to be written in more than one way. Finding Stanley decompositions in the semisimple case has only been addressed in a few cases. The semisimple Hamiltonian $(1, 1, \ldots, 1)$ resonance is studied in [5], and the (p, q, r) Hamiltonian resonance is treated in [29, Thm. 10.7.1]. The (p, q) resonance for two coupled nonlinear oscillators, not necessarily Hamiltonian, is studied in [22, §4.5].

Cushman and Sanders [10] solve the strong description problem for nilpotent N in the $\mathfrak{sl}_2$ style for $n \leq 6$ (except for the 6×6 nilpotent matrix having only one Jordan block), and also for any even n if all Jordan blocks of N are 2×2. Their method, which is now unnecessarily complicated, is based on the fact that vector fields $\mathbf{v}$ in $\mathfrak{sl}_2$ normal form are equivariant with respect to a certain one-parameter group; the method applies in principle to any nilpotent N, except that the calculations become too large.

Since the publication of [10], progress has been made in two directions: **1.** In [21], a method called **boosting** was given that obtains a description of the equivariant vector fields $\mathbf{v}$ from the