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A nonlinear parabolic equation with discontinuity in the highest order and applications

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Abstract

In this paper we establish a viscosity solution theory for a class of nonlinear parabolic equations with discontinuities of the sign function type in the second derivatives of the unknown function. We modify the definition of classical viscosity solutions and show uniqueness and existence of the solutions. These results are related to the limit behavior for the motion of a curve by a very small power of its curvature, which has applications in image processing. We also discuss the relation between our equation and the total variation flow in one space dimension.

MSC: 35D40; 35K67; 49L25; 53C44

Keywords: Curvature flow equations; Viscosity solutions; Discontinuous equations; Total variation flow

1. Introduction

We study a class of fully nonlinear parabolic equations with a jump discontinuity in the second derivatives of the unknown. The general equation form of equations we discuss is

$$u_t + F(\nabla u, \operatorname{sgn}(f(\nabla^2 u))) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty) \tag{1.1}$$

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with initial condition

$$u(x,0) = u_0(x) \text{ for all } x \in \mathbb{T}^n, \tag{1.2}$$

where $\mathbb{T}^n$ denotes the n-dimensional torus, F and f are assumed to be continuous functions and satisfy the ellipticity and u_0 is a continuous function on $\mathbb{T}^n$. More detailed assumptions will be given later. The function sgn is formally understood as the usual sign function:

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

We are particularly interested in the motion of a one dimensional graph with its normal velocity equal to the sign of its curvature:

$$u_t - \sqrt{1 + u_x^2}\, \operatorname{sgn}(u_{xx}) = 0. \tag{1.3}$$

It is not clear how one should handle such a discontinuity caused by the sign function of the second derivatives to obtain a unique solution of such equations. The classical theory of viscosity solutions (e.g., [11]) does not apply directly. In this work, we give a definition of viscosity solutions for (1.1) and show the uniqueness and existence of continuous solutions.

1.1. Motivations

Our problem is closely related to the mathematical models for image processing. It is well-known that the following nonlinear equation in two space dimensions has important applications in image denoising [1,8]:

$$u_t - |\nabla u| \left| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \right|^{\alpha - 1} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0. \tag{1.4}$$

This equation in two dimensions gives a level-set formulation of the motion of a curve Γ governed by the law:

$$V = \kappa^\alpha,$$

where V denotes the normal velocity and κ denotes the curvature of Γ; see [28–30] for results related to this geometric motion.

The choice of the exponent $\alpha > 0$ reflects a particular purpose for practical use in image processing. For the purpose of shape analysis, we need to pick a small α to cancel the pixel effect as fast as possible; on the other hand, if our aim is image denoising, we may want to remove small details while keeping main features unchanged, for which a large α seems more suitable. We are particularly interested in the behavior of the operator when $\alpha \to 0$, which formally comes to the equation:

$$u_t - |\nabla u| \operatorname{sgn}\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) = 0. \tag{1.5}$$