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Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates

Genni Fragnelli

Dipartimento di Matematica, Università di Bari "Aldo Moro", Via E. Orabona 4, 70125 Bari, Italy

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Abstract

We consider non-smooth general degenerate/singular parabolic equations in non-divergence form with degeneracy and singularity occurring in the interior of the spatial domain, in presence of Dirichlet or Neumann boundary conditions. In particular, we consider well posedness of the problem and then we prove Carleman estimates for the associated adjoint problem.

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E-mail address: genni.fragnelli@uniba.it.

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1. Introduction

The present paper is devoted to give a full analysis of the following problem:

$$\begin{cases} u_t - a(x)u_{xx} - \dfrac{\lambda}{b(x)}u = h(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ Bu(0) = Bu(1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $Bu(x) = u(t,x)$ or $Bu(x) = u_x(t,x)$ for all $t \in [0,T]$, $Q_T := (0,T) \times (0,1)$, χ_ω is the characteristic function of a set $\omega \subset (0,1)$, $u_0 \in L^2_{\frac{1}{a}}(0,1)$ and $h \in L^2_{\frac{1}{a}}(Q_T) := L^2(0,T; L^2_{\frac{1}{a}}(0,1))$. Here $L^2_{\frac{1}{a}}(0,1)$ is the Hilbert space

$$L^2_{\frac{1}{a}}(0,1) := \left\{ u \in L^2(0,1) \mid \int_0^1 \frac{u^2}{a} dx < \infty \right\},$$

endowed with the inner product

$$\langle u, v \rangle^2_{L^2_{\frac{1}{a}}(0,1)} := \int_0^1 \frac{uv}{a} dx, \qquad \text{for every } u, v \in L^2_{\frac{1}{a}}(0,1),$$

which induces the obvious associated norm.

Moreover, we assume that the constant λ satisfies suitable assumptions described below and the functions a and b, that can be *non-smooth*, degenerate at the same interior point $x_0 \in (0,1)$ that can belong to the control set ω. The fact that both a and b degenerate at x_0 is just for the sake of simplicity and shortness: all the stated results are still valid if they degenerate at different points. We shall admit different types of degeneracy for a and b. In particular, we make the following assumptions:

Hypothesis 1.1. Double weakly degenerate case (WWD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a, b \in W^{1,1}(0,1)$ and there exist $K_1, K_2 \in (0,1)$ such that $(x - x_0)a' \le K_1 a$ and $(x - x_0)b' \le K_2 b$ a.e. in $[0,1]$.

Hypothesis 1.2. Double strongly degenerate case (SSD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a, b \in W^{1,\infty}(0,1)$ and there exist $K_1, K_2 \in [1,2)$ such that $(x - x_0)a' \le K_1 a$ and $(x - x_0)b' \le K_2 b$ a.e. in $[0,1]$.

Hypothesis 1.3. Weakly strongly degenerate case (WSD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a \in W^{1,1}(0,1)$, $b \in W^{1,\infty}(0,1)$ and there exist $K_1 \in (0,1)$, $K_2 \in [1,2)$ such that $(x - x_0)a' \le K_1 a$ and $(x - x_0)b' \le K_2 b$ a.e. in $[0,1]$.

Hypothesis 1.4. Strongly weakly degenerate case (SWD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0,1)$, $b \in W^{1,1}(0,1)$, and there exist $K_1 \in [1,2)$, $K_2 \in (0,1)$ such that $(x - x_0)a' \le K_1 a$ and $(x - x_0)b' \le K_2 b$ a.e. in $[0,1]$.