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The complex viewpoint for transverse impasse points of quasi-linear differential equations

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Abstract

The main purpose of this paper is to show that the singularities called *impasse points* of real implicit ordinary differential equations are generically branch points of the solution, using a complex viewpoint. This research starts from a result asserting that the real solution in most cases behaves like $\pm\sqrt{x - x_0}$ at an impasse point (x_0, y_0). We extend the notion of regular impasse point defined in previous works, and consider instead transverse impasse point, where the underlying vector field is transverse to the singular locus, even in the case when this hypersurface is not a manifold locally.

Here we make use of Puiseux expansions and we show that, under generic hypotheses, the solution is multivalued at such a point. We prove that the Puiseux exponent is related simply to the multiplicity of the impasse point in the singular locus: if M is the total multiplicity of the singularity (z_0, y_0), there is a unique solution at this point and it is $M + 1$-valued.

Keywords: Complex implicit differential equations; Non-autonomous systems; Singularity; Puiseux expansion; Impasse point; DAEs

1. Introduction

Let $n > 0$ be an integer. n-dimensional implicit systems of real quasi-linear ODEs can be written:

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$$A(x, y)\frac{dy}{dx} = b(x, y) \tag{1}$$

where $A(x, y)$ (resp. $b(x, y)$) is a square matrix (resp. an n-vector) supposed to be analytic where defined.

A singular point of (1) is any value (x_0, y_0) such that $\det A(x_0, y_0) = 0$.

Under generic hypotheses a solution $x \mapsto y(x)$ can reach such a singular point within a finite interval, and then not be defined anymore when $x > x_0$. This never occurs for the solutions of explicit system of ODEs, i.e. when $\text{rank} A(x, y) = n$ for all the values of x and y. This is the reason why these singularities have been usually named impasse points [3,13].

This phenomenon does not have any counterpart in the complex field: one cannot give an immediate meaning to the expression *is not defined anymore*. Our aim is to find the nature of the solutions when using the complex viewpoint.

Such considerations on impasse points arose in the context of non-linear Differential-Algebraic Equations (DAEs), which generalize the case of implicit ODEs [18,3,4,13,20]. However the complex viewpoint has not been used to our knowledge.

Example 1.1. The real 2-dimensional example

$$y_1^2 + y_2 = 0, \frac{dy_2}{dx} = 0, y(0) = (1; -1)$$

from [13, p. 133] shows the interest of the complex resolution. There is a unique real solution to this Initial Condition problem: $y(x) = (\sqrt{1-x}, x-1)$. When $x > 1$, $1 - x < 0$ thus y_1 is not defined after $x = 1$: this is a typical forward impasse point. In the complex space, where we replace x with z, the solution reads also $y_1(z) = \sqrt{1-z}$ and is 2-valued. $z \mapsto y_1(z)$ is defined over $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \{1\}$. It has an algebraic branch point at $z_0 = 1$. The real behavior can be obtained by observation of the real part of the complex solution.

Our main result generalizes the theorems from ([11], [17] p. 569), which assert that under generic hypotheses, the solution at a regular impasse point of an implicit ODE is equivalent to $\pm\sqrt{x - x_0}$, locally around the singular point. We have extended these generic conditions to handle a wider class of singularities, called transverse impasse points. At such points, we prove the uniqueness of a multivalued complex solution, with a ramification index related to the multiplicity (see Subsection 2.3) of the point in the singular locus. We make use of a convergent Puiseux expansion [21] to develop the solution at the singular point, after parameterization of the derivation variable z.

This is the subject of Section 2, while in the third one we apply the results to the real case and expose examples.

2. The complex viewpoint

2.1. Some notations

We work in the complex space $\mathbb{C} \times \mathbb{C}^n$ and the current point is noted $X = (z, y)$ where $z \in \mathbb{C}$ is the derivative variable. For any map g, analytic from an open subset of $\mathbb{C} \times \mathbb{C}^n$ to $\mathbb{C}$, we denote

$$V(g) = \{(z, y) \in \mathbb{C} \times \mathbb{C}^n / g(z, y) = 0\}.$$