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Algebraic differential equations with functional coefficients concerning ζ and Γ

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Abstract

We prove that the Riemann zeta function and the Euler gamma function cannot satisfy a class of algebraic differential equations with functional coefficients that are connected to the zeros of the Riemann zeta function on the critical line.

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The famous, as yet unproven, Riemann hypothesis states that all the nontrivial zeros of the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ (analytically continued as a meromorphic function in $\mathbf{C}$) lie on the critical line $\mathbf{L} := \{z \in \mathbf{C} : \text{Re}z = 1/2\}$. Although every effort to prove the Riemann hypothesis has failed, it is however well-known that ζ has an infinity of zeros on the critical line (see e.g. [15], p. 256). In this paper, we will prove that the Riemann zeta function ζ and the Euler gamma function Γ, which are connected by the well-known Riemann functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z)\zeta(1-z),$$

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cannot satisfy a class of algebraic differential equations with functional coefficients satisfying a growth condition connected to infinitely many zeros of ζ on the critical line $\mathbf{L}$.

In a profound theorem Hölder [4] proved that the Euler gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re} z > 0$$

(analytically continued as a meromorphic function in the complex plane $\mathbf{C}$) does not satisfy any nontrivial algebraic differential equation with polynomial coefficients, i.e., if $P(w_0, w_1, \cdots, w_n)$ is a nonzero polynomial in its arguments with coefficients being polynomials in $\mathbf{C}$, then

$$P(\Gamma, \Gamma' \cdots, \Gamma^{(n)})(z) \not\equiv 0$$

where $z \in \mathbf{C}$. In Problem 18 of his famous list of 23 problems, Hilbert raised the question whether the Riemann zeta function ζ and allied functions satisfy any nontrivial algebraic differential equations (see [3]). Hilbert himself stated that ζ does not satisfy any nontrivial algebraic differential equation, and the problem was solved in great generality by Ostrowski [12]. Since then, these results have been extended in many different directions (see [1,2,5,7–9,11,13,16], etc. for related results and references therein), for example by also considering more general functional coefficients in considered equations. In contrast to the functional relation of ζ and Γ by the above mentioned Riemann functional equation, one is naturally interested in knowing whether ζ and Γ are related by any nontrivial algebraic differential equation. Bank and Kaufman [1] proved that Γ cannot satisfy any nontrivial algebraic differential equation with coefficients in a class of meromorphic functions, which however grow more slowly than ζ (see the precise statement in [1] and [6] in terms of the Nevanlinna characteristic function, which will not be used in the present paper); thus this result cannot apply to algebraic differential equations whose coefficients involving ζ. Voronin [16] proved that ζ cannot satisfy a large class of differential equations; but the result does not include the case when coefficients of algebraic differential equations involve Γ. More recently, Markus [11] showed that the composition function $\zeta(\sin(2\pi z))$ does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives, and conjectured that the Riemann zeta function ζ itself does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives, or equivalently (see [11]), Γ does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of ζ and its derivatives. In the present paper we will show that ζ and Γ do not satisfy a class of algebraic differential equations with functional coefficients in a "wild" class $\mathcal{L}$ of complex functions (not necessarily meromorphic or even continuous) that are connected to the zeros of ζ on the critical line, which particularly contains the ring of polynomials. The main theorem is shown to be best possible in a certain sense and has a number of consequences, one of which particularly shows that $P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)) \not\equiv 0$ in $\mathbf{C}$ for any nontrivial distinguished polynomial P whose coefficients can be allowed to be any polynomials of ζ over $\mathbf{C}$, over the ring of polynomials or, more generally, over the class $\mathcal{L}$ (see Corollary 3 below).

As mentioned above, the Riemann zeta function ζ has infinitely many zeros on the critical line $\mathbf{L}$. In fact, by the Riemann functional equation, ζ has infinitely many zeros on the half critical lines $\{z \in \mathbf{C} : \mathrm{Re} z = 1/2, \mathrm{Im} z > 0\}$ and $\{z \in \mathbf{C} : \mathrm{Re} z = 1/2, \mathrm{Im} z < 0\}$.