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Localization and vector spherical harmonics

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Abstract

This paper establishes the following localization property for vector spherical harmonics: a wide class of non-local, vector-valued operators reduce to local, multiplication-type operations when applied to a vector spherical harmonic. As localization occurs in a very precise, quantifiable and explicitly computable fashion, the localization property provides a set of useful formulae for analyzing vector-valued fractional diffusion and non-local differential equations defined on $\mathcal{S}^{d-1}$. As such analyses require a detailed understanding of operators for which localization occurs, we provide several applications of the result in the context of non-local differential equations.

Keywords: Dynamical systems; Non-local equations; Spectral theory; Special functions

1. Introduction

Spherical harmonics provide an indispensable tool for conducting harmonic analysis over the $(d-1)$-sphere [1–5]. For instance, formulations of classical concepts such as the Fourier transform, Sobolev spaces or fractional diffusion operators on $\mathcal{S}^{d-1}$ typically invoke spherical harmonics in some form or fashion. Many well-known relations that hold between functions $f(\mathbf{x})$ on Euclidean space $\mathbb{R}^d$ and their Fourier transforms $\hat{f}(\xi)$ therefore have natural analogues expressed in terms of spherical harmonics. The classical Funk–Hecke formula [2,4] provides an elementary but important and useful example along these lines: Convolution-type operators of the form

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$$g(\boldsymbol{\omega}) \xrightarrow{f*g} \int\limits_{\mathcal{S}^{d-1}} f\left(|\boldsymbol{\omega}-\boldsymbol{\sigma}|\right) g(\boldsymbol{\sigma})\, \mathrm{d}\mathcal{S}^{d-1}_{\boldsymbol{\sigma}} \qquad (\boldsymbol{\omega},\boldsymbol{\sigma} \in \mathcal{S}^{d-1}) \tag{1}$$

reduce to multiplication operators on expansion coefficients when expressed in terms of spherical harmonics. In particular, if $g(\boldsymbol{\omega})$ coincides with a singleton spherical harmonic then the convolution (1) reduces to scalar multiplication.

The Funk–Hecke theorem therefore provides a natural tool for studying non-local differential equations and related problems posed on the $(d-1)$-sphere. For example, just as the non-local fractional diffusion operator

$$(-\Delta)^{\frac{s}{2}} g(\mathbf{x}) := C_{s,d}\left(\text{p.v.} \int\limits_{\mathbb{R}^d} \frac{g(\mathbf{x})-g(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{d+s}}\, \mathrm{d}z\right) \quad \text{for} \quad C_{s,d} := \frac{s2^{s-1}\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(1-\frac{s}{2})}$$

on $\mathbb{R}^d$ admits a simple, local description $\hat{g}(\xi) \to |\xi|^s \hat{g}(\xi)$ via the Fourier transform [6,7], the Funk–Hecke formula shows that the analogous operator

$$(-\Delta_{\mathcal{S}^{d-1}})^{\frac{s}{2}} g(\boldsymbol{\omega}) := \left(\text{p.v.} \int\limits_{\mathcal{S}^{d-1}} \frac{g(\boldsymbol{\omega})-g(\boldsymbol{\sigma})}{|\boldsymbol{\omega}-\boldsymbol{\sigma}|^{(d-1)+s}}\, \mathrm{d}\mathcal{S}^{d-1}_{\boldsymbol{\sigma}}\right) \tag{2}$$

acts locally on spherical harmonics in a similar way. Indeed, $(-\Delta_{\mathcal{S}^{d-1}})^{\frac{s}{2}}$ acts via $\hat{g}_\ell \to \lambda_\ell \hat{g}_\ell$ for $\lambda_\ell \sim C\ell^s$ some positive sequence of scalars. Other related concepts, such as fractional-order Sobolev spaces on $\mathcal{S}^{d-1}$, can also be defined and analyzed using this translation between non-local operators and Fourier-like techniques [8,4].

Motivated by recent interest in non-local differential equations defined over $\mathcal{S}^{d-1}$ [9–18], as well as by the recent interest in non-linear fractional diffusion equations [19–24], we establish a broad generalization of the Funk–Hecke theorem that applies to a wide range of non-local operators acting on vector-valued functions. The genesis of this result lies in a recent body of work that studies non-linear, non-local equations of the form

$$\partial_t \Phi = F(\Phi) \qquad (\Phi : \mathcal{S}^{d-1} \to \mathbb{R}^d)$$

by using stability theory and classical dynamical systems techniques. A proper execution of the dynamical systems approach to such equations necessitates an in-depth characterization of linear evolution equations $\partial_t \Psi = L[\Psi]$, where the operator L acts non-locally

$$\Psi(\boldsymbol{\omega}) \xrightarrow{L} \int\limits_{\mathcal{S}^{d-1}} f\left(|\boldsymbol{\omega}-\boldsymbol{\sigma}|\right) k(\boldsymbol{\omega},\boldsymbol{\sigma})\Psi(\boldsymbol{\sigma})\, \mathrm{d}\mathcal{S}^{d-1}_{\boldsymbol{\sigma}} \qquad (\boldsymbol{\omega},\boldsymbol{\sigma} \in \mathcal{S}^{d-1},\ k(\boldsymbol{\omega},\boldsymbol{\sigma}) \in \mathbb{R}^{d\times d}) \tag{3}$$

on vector-valued functions. The obvious similarity between (1)–(2) and (3) brings to the forefront the question of whether a useful analogue of the Funk–Hecke formula exists in the vector-valued setting, and if so, the secondary task of uncovering a sufficiently broad class of non-local operators (3) to which such a generalization applies.