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On a generalization of the Miranda Theorem and its application to boundary value problems

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Abstract

In this paper we have proved a multi-valued version of the Miranda Theorem involving an admissible map (in the sense of Górniewicz). Examples of its application to boundary value problems have been given.

Keywords: Nonlinear boundary value problem; Admissible map; Decomposable maps; R_δ-sets; The Miranda Theorem

1. Introduction

The theorems of Poincaré–Miranda type are straightforward generalizations of the well-know Bolzano's existence theorem (proved in 1817) which states:

Theorem 1. *Let $F : [-M, M] \to \mathbb{R}$ be a continuous function and $F(-M)F(M) < 0$, then for some $x \in (-M, M)$, it holds $F(x) = 0$.*

The following generalized result of the Bolzano Theorem was announced without the proof by Poincaré in 1886, [14]:

Theorem 2. *Let $F : [-M, M]^k \to \mathbb{R}^k$, $F = (F_1, \dots, F_k)$, be a continuous function, satisfying the following conditions:*

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$$F_i(x_1, \ldots, x_{i-1}, M, x_{i+1}, \ldots, x_k) \geq 0$$

and

$$F_i(x_1, \ldots, x_{i-1}, -M, x_{i+1}, \ldots, x_k) \leq 0$$

for $i = 1, \ldots, k$. *Then there exists* $x \in [-M, M]^k$ *such that* $F(x) = 0$.

Theorem of Poincaré was rediscovered by Miranda, who in 1940 showed that it was equivalent to Brouwer's fixed point theorem, and has come to be known as the Miranda Theorem, [9].

Recently, the following generalization of Theorem 2 has been proved, [18]:

Theorem 3. *Let* F *be an upper semicontinuous map from the hypercube* $[-M, M]^k$ *into convex and compact subsets of* $\mathbb{R}^k$ *and satisfying for* $d = (d_1, \ldots, d_k) \in \mathbb{R}^k$ *the following conditions*

$$\text{if} \quad d \in F(x_1, \ldots, x_{i-1}, M, x_{i+1}, \ldots, x_k), \quad \text{then } d_i \geq 0$$

and

$$\text{if} \quad d \in F(x_1, \ldots, x_{i-1}, -M, x_{i+1}, \ldots, x_k), \quad \text{then } d_i \leq 0,$$

for every $i = 1, \ldots, k$. *Then there exists* $x \in [-M, M]^k$ *such that* $0 \in F(x)$.

In this paper we have proved a generalization of Theorem 3 (see Theorem 5). Moreover, we have shown that the new version is very useful in the theory of ordinary differential equations.

Most nonlinear differential, integral or, more generally, functional equations have the form $Lx = N(x)$, where L is a linear and N nonlinear operator, in appropriate Banach spaces. We have no problem if L is a linear Fredholm operator of index 0. Then the kernel of the linear part of the above equation is trivial. It means that there exists an integral operator and we can apply known topological methods to prove the existence theorems. If kernel L is nontrivial then the equation is called resonant and one can manage the problem by using the coincidence degree in that case [8]. But, if the domain is unbounded (for example the half-line) the operator is usually non-Fredholm. Such problems have been studied by different methods in many papers. We mention only [3,8,12,15,16].

The method presented in this paper can be applied to both non-resonant and resonant problems. Furthermore, one can use this method for any boundary value problem (we write BVP) posed on interval or on the half-line (compare examples).

Using the generalized Miranda Theorem we have proved theorems about existence for systems of k equations $x'' = f(t, x, x')$, where $f : [0, 1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is a vector function, subject to various boundary conditions. We have considered a BVP with boundary conditions $x'(0) = 0$ and $x(1) = 0$ and a problem on the half-line with the boundary conditions $x'(0) = 0$, $x'(\infty) = 0$ (such problems can be found, for instance, in the papers [1,19]). Moreover, we have got an existence result for the Dirichlet BVP (such problems have been considered in many papers, see for instance [2,17,22]). Finally, we would like to pay a special attention to the results for the Neumann BVP. There are many papers which considered this problem (see [7,10,11,20,21] and the references therein). The problems discussed in the above-mentioned papers are scalar and functions on the right-hand side of the equation have specific forms. In the best of our knowledge