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A determining form for the damped driven nonlinear Schrödinger equation—Fourier modes case

Michael S. Jolly [a,*], Tural Sadigov [a], Edriss S. Titi [b,c]

[a] *Department of Mathematics, Indiana University, Bloomington, Bloomington, IN 47405, USA*

[b] *The Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel*

[c] *Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA*

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Abstract

In this paper we show that the global attractor of the 1D damped, driven, nonlinear Schrödinger equation (NLS) is embedded in the long-time dynamics of a determining form. The determining form is an ordinary differential equation in a space of trajectories $X = C_b^1(\mathbb{R}, P_m H^2)$ where P_m is the L^2-projector onto the span of the first m Fourier modes. There is a one-to-one identification with the trajectories in the global attractor of the NLS and the steady states of the determining form. We also give an improved estimate for the number of the determining modes.

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[*] Corresponding author.

E-mail addresses: msjolly@indiana.edu (M.S. Jolly), tsadigov@indiana.edu (T. Sadigov), etiti@math.uci.edu, edriss.titi@weizmann.ac.il (E.S. Titi).

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1. Introduction

The damped, driven, nonlinear Schrödinger equation (NLS), (2.1), has been derived in various areas of physics, and widely investigated (see e.g. [4] and references therein). In plasma physics, the NLS is a model for the propagation of an intense laser beam through a nonlinear medium (see e.g. [7]). In this model the unknown function $u(x,t)$ is the electrical field amplitude, t is the distance in the direction of the propagation and x is a transverse spatial variable. Absorption of the electromagnetic wave by the medium is accounted for by linear damping. Some resonant forcing of small amplitude (for example, a traveling wave) is used to compensate weak dissipative losses (i.e., absorption). The NLS also describes the single particle properties of Bose–Einstein condensate (BEC) (see e.g. [3]), in which a gas of bosons is cooled to very low temperatures. In this case, known as the Gross–Pitaevski equation, $u(x,t)$ describes the macroscopic wave function of the condensate; t is time and x is a spatial variable. A constant damping rate (absorption) γ describes inelastic collisions with the background gas which occur when the particle density is very large. The forcing term represents the interatomic forces of the condensate. We note that the NLS is also investigated in deep-water phenomena and in the collapse of Langmuir waves (see e.g. [7]). In this paper, we consider a force general enough to include the above applications where it can be periodic in space and independent of time.

The undamped, unforced case has been extensively studied in modern mathematical physics (see e.g. [5]). Well-posedness of (2.1), for nonzero forcing and $\gamma > 0$ is established by Ghidaglia in [12], where, under the assumption that the force is either time independent or time periodic, it is also proved that there exists a weak attractor in the Sobolev spaces H^1 and H^2. Later, it is proved in [18] that this weak attractor is in fact a global attractor in H^2 in the strong sense. In [13], assuming the force is smooth enough and periodic in spatial variable, Goubet proved that the global attractor $\mathcal{A}$ is smooth, meaning it is included and bounded in H^k, for any $k \geq 1$. This implies that $\mathcal{A}$ is in C^∞ due to classical Sobolev embeddings theorems. Finally in [16], it is proved that $\mathcal{A}$ is in fact contained in a subclass of the space of real analytical functions provided that the forcing term is real analytic. The long-time dynamics of the damped, driven NLS is entirely contained in the *global attractor* $\mathcal{A}$, a compact finite-dimensional set within the infinite-dimensional phase space H^k for any $k \geq 1$ (see [13]). It is shown in [16], for real analytical forcing, that the solutions on the attractor of the NLS are determined uniquely by their nodal values on only two sufficiently close nodes.

The finite dimensionality for the NLS can be stated more explicitly. It is also known that solutions of the NLS in $\mathcal{A}$ are determined by the asymptotic behavior of a sufficient finite number of Fourier modes (see [13,14]). To be precise, this means that if two complete trajectories in the global attractor coincide under the projection P_m onto a sufficiently large number, m, of low modes, then they are the same trajectory. These m-modes are called *determining modes* (see [11]). This notion of determining modes was used in [8] to find a *determining form* for the 2D Navier–Stokes equations (NSE). In [8], the determining form is an ordinary differential equation in an *infinite dimensional* Banach space $X = C_b(\mathbb{R}, P_m H)$, governing the evolution of trajectories. Here H is a Hilbert space which is a natural phase space for the 2D NSE (see [6,17]). The trajectories in the attractor of the 2D NSE are identified with traveling wave solutions of the determining form in [8].

A determining form of a different sort was found in [9] for the 2D NSE. It is based on *data assimilation by feedback control* through a general interpolant operator. It is general in the sense that it can be induced by a variety of determining parameters such as determining modes, nodal values and finite volumes (see [10], [15] and references therein). The steady states of this deter-