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Lie–Hamilton systems on the plane: Properties, classification and applications

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Abstract

We study Lie–Hamilton systems on the plane, i.e. systems of first-order differential equations describing the integral curves of a t-dependent vector field taking values in a finite-dimensional real Lie algebra of planar Hamiltonian vector fields with respect to a Poisson structure. We start with the local classification of finite-dimensional real Lie algebras of vector fields on the plane obtained in González-López, Kamran, and Olver (1992) [23] and we interpret their results as a local classification of Lie systems. By determining which of these real Lie algebras consist of Hamiltonian vector fields relative to a Poisson structure, we provide the complete local classification of Lie–Hamilton systems on the plane. We present and study through our results new Lie–Hamilton systems of interest which are used to investigate relevant non-autonomous differential equations, e.g. we get explicit local diffeomorphisms between such systems. We also analyse biomathematical models, the Milne–Pinney equations, second-order Kummer–Schwarz equations, complex Riccati equations and Buchdahl equations.

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1. Introduction

The relevance of nonautonomous differential equations is undoubtable both from the mathematical viewpoint and also from their overwhelming applications. In this work we will get a deeper insight into a particular class of systems of differential equations, the so-called Lie systems, which have drawn some attention during the past recent years due to their geometric properties and applications. For instance, the general solution for a Lie system can be obtained in terms of a superposition rule (see [13] and references therein).

More explicitly, a *Lie system* is a system of first-order differential equations describing the integral curves of a t-dependent vector field taking values in a finite-dimensional real Lie algebra of vector fields, a *Vessiot–Guldberg Lie algebra* [13,30]. Vessiot–Guldberg Lie algebras determine the main properties of Lie systems, e.g. Lie systems related to a solvable Vessiot–Guldberg Lie algebra of right-invariant vector fields on a Lie group are integrable [12]. Although Lie systems are a quite restricted class of differential equations [13,25], very recurrent systems appearing in the literature, e.g. most types of Riccati and Kummer–Schwarz equations, can be studied through these systems [9,40]. In this paper, we aim to study *Lie–Hamilton systems* [1,4,13,15], which form a relevant subclass of Lie systems. Our concern in them relies on their frequent appearance in classical mechanics and their special characteristics: integrability, symmetries and superposition rules [1,4,10,32].

A natural problem in the theory of Lie systems is the classification of Lie systems on a fixed manifold, which amounts to classifying finite-dimensional Lie algebras of vector fields on it. Lie accomplished the local classification of finite-dimensional real Lie algebras of vector fields on the real line [29]. More precisely, he showed that each such a Lie algebra is locally diffeomorphic to a Lie subalgebra of $\langle \partial_x, x\partial_x, x^2\partial_x \rangle \simeq \mathfrak{sl}(2)$ on a neighbourhood of each *generic point* x_0 of the Lie algebra [23,29]. He also performed the local classification of finite-dimensional Lie algebras of vector fields on $\mathbb{C}$ over the complex numbers [29] and, by an ingenious geometric argument and the previous result [23], the classification of finite-dimensional Lie algebras of vector fields on $\mathbb{R}^2$ over the reals in [31, p. 360].

Lie's local classification on the plane presented some unclear points which were misunderstood by several authors during the following decades. Later on, A. González-López, N. Kamran and P.J. Olver retook the problem and provided a clearer insight in [23]. Precisely, they proved in a modern geometric manner that every non-zero Lie algebra of vector fields on the plane is locally diffeomorphic around each generic point to one of the finite-dimensional real Lie algebras given in Section 3 of this work. For simplicity, we refer to this result as the *GKO classification*.

As every Vessiot–Guldberg Lie algebra on the plane is locally diffeomorphic around a generic point to a Lie algebra of the GKO classification, every Lie system on the plane is locally diffeomorphic to a Lie system taking values in a Vessiot–Guldberg Lie algebra within the GKO classification. So, the local properties of all Lie systems on the plane can be studied through the Lie systems related to the GKO classification. As a consequence, we say that the GKO classification gives the local classification of Lie systems on the plane.

The *minimal Lie algebra* of a Lie system is its smallest Vessiot–Guldberg Lie algebra [13]. In this paper we analyse the general properties of minimal Lie algebras of Lie–Hamilton systems on the plane. We demonstrate that they are, around generic points, Lie algebras of Hamiltonian vector fields with respect to a symplectic structure. We also provide several results allowing us to determine their algebraic structure.

It is known that each Lie–Hamilton system on a manifold N gives rise to a t-dependent Hamiltonian $h : (t, x) \in \mathbb{R} \times N \mapsto h_t(x) \in N$ whose functions $\{h_t\}_{t\in\mathbb{R}}$ and their successive Lie