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Probabilistic representations of solutions of elliptic boundary value problem and non-symmetric semigroups

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Abstract

In this paper, we use a probabilistic approach to show that there exists a unique, bounded continuous solution to the Dirichlet boundary value problem for a general class of second order non-symmetric elliptic operators L with singular coefficients, which does not necessarily have the maximum principle. The theory of Dirichlet forms and heat kernel estimates play a crucial role in our approach. A probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t\geq 0}$ generated by L is also given.

Keywords: Dirichlet boundary value problem; Singular coefficient; Non-symmetric semigroup; Probabilistic representation; Dirichlet form; Heat kernel estimate

1. Introduction and the main theorem

In this paper, we will use probabilistic methods to study the Dirichlet boundary value problem for second order elliptic differential operators:

$$\begin{cases} Lu = 0 & \text{in } D \\ u = f & \text{on } \partial D, \end{cases} \tag{1.1}$$

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where D is a bounded connected open subset of $\mathbb{R}^d$. The operator L is given by

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) - \operatorname{div}\hat{b}(x))u, \tag{1.2}$$

where $A(x) = (a_{ij}(x))_{i,j=1}^d$ is a Borel measurable, (not necessarily symmetric) matrix-valued function on D satisfying

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \quad \text{for any } \xi = (\xi_i)_{i=1}^d \in \mathbb{R}^d, x \in D \tag{1.3}$$

and

$$|a_{ij}(x)| \le \frac{1}{\lambda} \quad \text{for any } x \in D,\ 1 \le i, j \le d \tag{1.4}$$

for some constant $0 < \lambda \le 1$; $b = (b_1, \ldots, b_d)^*$ and $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_d)^*$ are Borel measurable $\mathbb{R}^d$-valued functions on D and c is a Borel measurable function on D satisfying $|b|^2 \in L^{p\vee 1}(D; dx)$, $|\hat{b}|^2 \in L^{p\vee 1}(D; dx)$ and $c \in L^{p\vee 1}(D; dx)$ for some constant $p > d/2$. Hereafter we use * to denote the transpose of a vector or matrix, and use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote respectively the standard norm and inner product of the Euclidean space $\mathbb{R}^d$.

In (1.1), $Lu = 0$ in D is understood in the distributional sense:

$$u \in H^{1,2}(D) \text{ and } \mathcal{E}(u, \phi) = 0 \text{ for every } \phi \in C_0^\infty(D),$$

where $H^{1,2}(D)$ is the Sobolev space on D with norm

$$\|f\|_{H^{1,2}} := \left(\int_D |\nabla f(x)|^2 dx + \int_D |f(x)|^2 dx\right)^{1/2},$$

$C_0^\infty(D)$ is the space of infinitely differentiable functions with compact support in D, and $(\mathcal{E}, D(\mathcal{E}))$ is the bilinear form associated with L:

$$\begin{aligned}\mathcal{E}(u, v) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^{d}\int_D b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &\quad - \sum_{i=1}^{d}\int_D \hat{b}_i(x)\frac{\partial(uv)}{\partial x_i}dx - \int_D c(x)u(x)v(x)dx, \\ D(\mathcal{E}) &= H_0^{1,2}(D)\end{aligned} \tag{1.5}$$

with $H_0^{1,2}(D)$ being the completion of $C_0^\infty(D)$ with respect to the Sobolev-norm $\|\cdot\|_{H^{1,2}}$. By setting $a = I$, $b = 0$, $\hat{b} = 0$ and $c = 0$ off D, we may assume that the operator L is defined on $\mathbb{R}^d$.