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An isoperimetric type inequality for the principal eigenvalue of Schrödinger operators depending on the curvature of a loop

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Dedicated to Nick Alikakos for his 60th birthday

Abstract

Let κ be the curvature of a smooth closed curve of length 2π. The 2π-periodic eigenvalue problems $u'' - g\kappa^2 u + \lambda u = 0$, where g is a real parameter, are considered and upper and lower bounds for their principal eigenvalues are obtained. The bounds are of isoperimetric type and characterize uniquely the curve in the case where equality occurs.

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1. Introduction

Let γ be a smooth closed curve in $\mathbb{R}^3$, of length 2π, parametrized by arclength s, and let $\kappa = \kappa(s)$ be the curvature of γ. We consider the associated Schrödinger operator

$$H(\gamma)u := -\frac{\mathrm{d}^2}{\mathrm{d}s^2}u(s) - \kappa^2 u(s)$$

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defined on the space of 2π-periodic, square integrable functions. We are concerned, in particular, in obtaining upper and lower bounds for the lowest eigenvalue of $H(\gamma)$.

The periodic eigenvalue problem associated with $H(\gamma)$

$$\begin{aligned} &u'' + \kappa^2 u + \lambda u = 0, \qquad 0 < s < 2\pi \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{1.1}$$

for γ being a simple, closed, smooth curve in the plane appeared in [1] in the context of the study of the motion of interfaces in the Cahn–Hilliard equation. The principal eigenvalue is negative, as can be seen by simply integrating the equation. It was conjectured by Alikakos and Fusco that the second eigenvalue is negative, too, unless the curve is a circle in which case it is equal to zero. This conjecture was proved in the affirmative in [10] for smooth curves in $\mathbb{R}^3$ while in [14] a local version was proved in the sense that if γ is a small perturbation of a circle but not an exact circle, then there are at least two negative eigenvalues. It has been also proved that the circle uniquely maximizes the lowest eigenvalue [6]. In this article we prove the following result

Theorem 1.1. *If $\lambda_0(\gamma)$ is the principal eigenvalue of (1.1) with κ being the curvature of a closed C^2 curve γ in $\mathbb{R}^3$ of length 2π, then*

$$-\frac{1}{2\pi}\int_0^{2\pi} \kappa^2\,\mathrm{d}s - K(\gamma) \le \lambda_0(\gamma) \le -\frac{1}{2\pi}\int_0^{2\pi}\kappa^2\,\mathrm{d}s, \tag{1.2}$$

where the constant $K(\gamma)$ is given by

$$K(\gamma) = \min_{r=1/2,1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\kappa^4\,\mathrm{d}s - \left(\frac{1}{2\pi}\int_0^{2\pi}\kappa^2\,\mathrm{d}s\right)^2\right\}^r. \tag{1.3}$$

Equality holds in the double inequality if and only if γ is a circle in which case $\kappa \equiv 1$ and $\lambda_0(\gamma) = -1$.

Since $-\int_0^{2\pi}\kappa^2\,\mathrm{d}s \le -2\pi$, an immediate consequence of (1.2) is that the principal eigenvalue of $H(\gamma)$ is maximized when γ is a circle. Another consequence, weaker than (1.2) but relevant to curves close to circles, is the estimate

$$-\frac{1}{2\pi}\int_0^{2\pi}\kappa^4\,\mathrm{d}s \le \lambda_0(\gamma) \le -\frac{1}{2\pi}\int_0^{2\pi}\kappa^2\,\mathrm{d}s \tag{1.4}$$

in which, again, equality occurs if and only if γ is a circle.

A more general class of operators

$$H(\gamma, g)u := -\frac{\mathrm{d}^2}{\mathrm{d}s^2}u(s) + g\kappa^2 u(s),$$