Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 133–148
Journal of Differential Equations
www.elsevier.com/locate/jde

Positive minimizers of the best constants and solutions to coupled critical quasilinear systems ☆

Dongsheng Kang

School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, PR China

Received 14 July 2014; revised 10 April 2015
Available online 9 September 2015

Abstract

In this paper, systems of quasilinear elliptic equations are investigated, which involve critical homogeneous nonlinearities and deferent Hardy-type terms. By variational methods and careful analysis, positive minimizers of the related best Sobolev constants are found and the existence of positive solutions to the systems is verified. The results are new even in the case $p = 2$.

MSC: 35J47; 35J62

Keywords: System; Quasilinear elliptic equation; Minimizer; Solution; Variational method

1. Introduction

In this paper, we mainly study the following system of quasilinear elliptic equations:

$$\begin{cases} -\Delta_p u - \mu_1 \dfrac{|u|^{p-2}u}{|x|^p} = \dfrac{\eta\alpha}{p^*}|u|^{\alpha-2}|v|^{\beta}u + |u|^{p^*-2}u + \dfrac{1}{p}Q'_s(u,v), \ \text{ in } \Omega, \\ -\Delta_p v - \mu_2 \dfrac{|v|^{p-2}v}{|x|^p} = \dfrac{\eta\beta}{p^*}|u|^{\alpha}|v|^{\beta-2}v + |v|^{p^*-2}v + \dfrac{1}{p}Q'_t(u,v), \ \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

☆ This work is supported by the Fundamental Research Funds for the Central Universities, South-Central University for Nationalities (CZW15053).
E-mail address: dongshengkang@scuec.edu.cn.

http://dx.doi.org/10.1016/j.jde.2015.08.042
0022-0396/

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary such that $0 \in \Omega$, $\Delta_p \cdot := \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ is the p-Laplace operator, $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, Q'_s, Q'_t are partial derivatives of the homogeneous C^1-function $Q(s,t)$:

$$Q(s,t) := a_1|s|^p + a_2 p|s|^{p-2}st + a_3 p|t|^{p-2}st + a_4|t|^p, \quad (s,t) \in \mathbb{R}^2, \quad p \ge 2,$$

and the parameters satisfy

$(\mathcal{H}_1)$ $1 < p < N$, $\eta > 0$, $0 \le \mu_2 \le \mu_1 < \bar{\mu} := (\frac{N-p}{p})^p$, $\alpha, \beta > 1$, $\alpha + \beta = p^*$.
$(\mathcal{H}_2)$ $a_i > 0$, $1 \le i \le 4$, and there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1(|u|^p + |v|^p) \le Q(u,v) \le \lambda_2(|u|^p + |v|^p), \quad \forall (u,v) \in W \times W.$$

Let $W := W_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to $(\int_\Omega |\nabla \cdot|^p \, dx)^{1/p}$. Energy functional of (1.1) is defined on the product space $W^2 := W \times W$ by

$$J(u,v) := \frac{1}{p} \int_\Omega \big(E(u,v) - Q(u,v)\big) dx - \frac{1}{p^*} \int_\Omega F(u,v) dx,$$

where

$$E(u,v) := |\nabla u|^p + |\nabla v|^p - \frac{\mu_1|u|^p + \mu_2|v|^p}{|x|^p},$$
$$F(u,v) := |u|^{p^*} + |v|^{p^*} + \eta|u|^\alpha|v|^\beta.$$

Then for all $p \ge 2$, $J \in C^1(W^2, \mathbb{R})$ and $(u,v) \in W^2$ is said to be a solution to (1.1) if

$$(u,v) \ne (0,0), \qquad \langle J'(u,v), (\varphi, \phi)\rangle = 0, \quad \forall (\varphi, \phi) \in W^2,$$

where $J'(u,v)$ denotes the Fréchet derivative of J at (u,v).

Problem (1.1) is related to the Hardy inequality ([15]):

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \le \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.2}$$

By the Hardy inequality, the operator $L = -(\Delta_p \cdot + \frac{\mu}{|x|^p}|\cdot|^{p-2}\cdot)$ is positive on W for all $\mu < \bar{\mu}$ and the first eigenvalue $\Lambda_1(\mu)$ of L on W is well defined.

Let $D := D^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla \cdot|^p \, dx)^{1/p}$. For all $\mu < \bar{\mu}$, by $(\mathcal{H}_1)$ and (1.2) the following best Sobolev-type constants are well defined and are crucial for the study of (1.1):

$$S(\mu) := \inf_{u \in D\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}\big) dx}{\big(\int_{\mathbb{R}^N} |u|^{p^*} dx\big)^{\frac{p}{p^*}}}, \tag{1.3}$$

$$S(\mu_1, \mu_2) := \inf_{(u,v) \in D^2 \setminus \{(0,0)\}} \frac{\int_{\mathbb{R}^N} E(u,v) dx}{\big(\int_{\mathbb{R}^N} F(u,v) dx\big)^{\frac{p}{p^*}}}. \tag{1.4}$$

By (1.3), for all $u \in D \setminus \{0\}$, testing (1.4) with $(u,0)$ we have $S(\mu_1, \mu_2) \le S(\mu_1)$.