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Boundedness, blowup and critical mass phenomenon in competing chemotaxis

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Abstract

We consider the following attraction–repulsion Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary. The system models the chemotactic interactions between one species (denoted by u) and two competing chemicals (denoted by v and w), which has important applications in Alzheimer's disease. Here all parameters χ, ξ, α, β, γ and δ are positive. By constructing a Lyapunov functional, we establish the global existence of uniformly-in-time bounded classical solutions with large initial data if the repulsion dominates or cancels attraction (i.e., $\xi\gamma\geq\alpha\chi$). If the attraction dominates (i.e., $\xi\gamma<\alpha\chi$), a critical mass phenomenon is found. Specifically speaking, we find a critical mass $m_*=\frac{4\pi}{\alpha\chi-\xi\gamma}$ such that the solution exists globally with uniform-in-time bound if $M<m_*$ and blows up if $M>m_*$ and $M\notin\{\frac{4\pi m}{\theta}: m\in\mathbb{N}^+\}$ where $\mathbb{N}^+$ denotes the set of positive integers and $M=\int_\Omega u_0dx$ the initial cell mass.

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1. Introduction

This paper is concerned with the initial–boundary value problem of the following attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ \tau_1 v(x,0) = \tau_1 v_0(x),\ \tau_2 w(x,0) = \tau_2 w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and ν denotes the outward normal vector of $\partial\Omega$. The model (1.1) was proposed in [28] to describe the aggregation of *Microglia* in the central nervous system in Alzheimer's disease due to the interaction of chemoattractant (β-amyloid) and chemorepellent (TNF-α), where $u(x,t)$, $v(x,t)$ and $w(x,t)$ in the model (1.1) denote the concentrations of *Microglia*, chemoattractant and chemorepellent which are produced by *Microglia*, respectively. The positive parameters χ and ξ are called the chemotactic coefficients, and $\chi,\beta,\gamma,\delta>0$ are chemical production and degradation rates. τ_1, τ_2 are constants equal to 0 or 1 justifying whether the change of chemicals is stationary or dynamical in time. The model (1.1) was also a particularized system introduced in the paper [33] to model the quorum sensing effect in the chemotactic movement.

Well-known as the Keller–Segel model (see [23]), the prototype of classical attractive chemotaxis model reads as

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v),\\ \tau_1 v_t = \Delta v + \alpha u - \beta v. \end{cases} \tag{1.2}$$

One prominent property of the Keller–Segel model (1.2) is the existence of a Lyapunov functional which continuously stimulates a vast amount of mathematical studies on various aspects of mathematics such as blowup, boundedness, traveling waves, pattern formations, critical mass phenomenon and critical sensitivity exponents (e.g. see [4,5,15,16,19,29,31,32,37,40,41] and the references therein, and review articles [13,18,39]).

On the other hand, for the classical repulsive chemotaxis model which reads as follows:

$$\begin{cases} u_t = \Delta u + \nabla\cdot(\xi u\nabla w),\\ \tau_2 w_t = \Delta w + \gamma u - \delta w, \end{cases}$$

a Lyapunov functional different from that of the attractive Keller–Segel model was found in [6], which led to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. The results on the repulsive Keller–Segel model are very limited and a further study on such model was recently given in [36].

Mathematically the three-component attraction–repulsion chemotaxis system (1.1) modeling the aggregation of *Microglia* is a coupled attractive and repulsive Keller–Segel model, and hence is referred to as the attraction–repulsion Keller–Segel (abbreviated as ARKS) model. It is hard