Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 197–217

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic study of the initial value problem to a standard one pressure model of multifluid flows in nondivergence form ☆

M. Colombeau

Instituto de Matemática e Estatistica, Universidade de São Paulo, Brazil

Received 24 February 2014; revised 11 May 2015

Available online 26 September 2015

Abstract

We construct families of approximate solutions to the initial value problem and provide complete mathematical proofs that they tend to satisfy the standard system of isothermal one pressure two-fluid flows in 1-D when the data are L^1 in densities and L^∞ in velocities. To this end, we use a method that reduces this system of PDEs to a family of systems of four ODEs in Banach spaces whose smooth solutions are these approximate solutions. This method is constructive: using standard numerical methods for ODEs one can easily and accurately compute these approximate solutions which, therefore, from the mathematical proof, can serve for comparison with numerical schemes. One observes agreement with previously known solutions from scientific computing (Evje and Flatten, 2003 [16]) We show that one recovers the solutions of these authors (exactly in one case, with a slight difference in another case). Then we propose an efficient numerical scheme for the original system of two-fluid flows and show it gives back exactly the same results as the theoretical solutions obtained above.

MSC: 35D30; 35F25; 65M06; 76-XX

Keywords: Partial differential equations; Approximate solutions; Weak asymptotic methods; Fluid dynamics

☆ This research has been done thanks to financial support of FAPESP, processo 2012/15780-9.
E-mail address: mcolombeau@ime.usp.br.

http://dx.doi.org/10.1016/j.jde.2015.08.039

1. Introduction

We study a basic model used to describe mathematically a mixture of two immiscible fluids in the isothermal case and without transfer of momentum between the two fluids, [16] p. 179, [9] p. 465,

$$\frac{\partial}{\partial t}(\rho_1\alpha_1) + \frac{\partial}{\partial x}(\rho_1\alpha_1 u_1) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_2\alpha_2) + \frac{\partial}{\partial x}(\rho_2\alpha_2 u_2) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho_1\alpha_1 u_1) + \frac{\partial}{\partial x}(\rho_1\alpha_1(u_1)^2) + \frac{\partial}{\partial x}((p_1 - p_1^{int})\alpha_1) + \alpha_1\frac{\partial}{\partial x}(p_1^{int}) = g\alpha_1\rho_1, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho_2\alpha_2 u_2) + \frac{\partial}{\partial x}(\rho_2\alpha_2(u_2)^2) + \frac{\partial}{\partial x}((p_2 - p_2^{int})\alpha_2) + \alpha_2\frac{\partial}{\partial x}(p_2^{int}) = g\alpha_2\rho_2, \tag{4}$$

$$\alpha_1 + \alpha_2 = 1, \tag{5}$$

$$p_1 = K_1\rho_1 - b_1, \quad p_2 = K_2\rho_2 - b_2, \tag{6}$$

where the two fluids are denoted by the indices 1 and 2, for instance mixture of oil and natural gas in extraction tubes of oil exploitation [2]. The physical variables are the densities $\rho_i(x,t)$, the velocities $u_i(x,t)$, the volumic proportions $\alpha_i(x,t)$, the pressures $p_i(x,t)$, the phasic pressures $p_i^{int}(x,t)$ at the interface, $i = 1, 2$, and g is the component of the gravitational acceleration in the direction of the tube. Equations (6) are the state laws stated in [16] p. 179; it is assumed $b_1 - b_2 > 0$, $K_1 > 0$ and $K_2 > 0$. Equations (1) and (2) are the continuity equations for each fluid: they express mass conservation. Equations (3) and (4) are the Euler equations for each fluid: they express momentum conservation. A natural assumption is to state the equality of the four pressures p_i and p_i^{int}, $i = 1, 2$. This simplest assumption of equal pressure leads to a nonhyperbolic model, called the equal pressure model [15] p. 677, [22] p. 2589, [25] p. 287, [26] pp. 372–373 that we study in this paper.

We construct families of differentiable functions $S(x,t,\epsilon)$ that, when plugged into the equal pressure model, tend asymptotically to satisfy it when $\epsilon \to 0$. We prove that these families of functions are weak asymptotic methods. The concept of weak asymptotic method and its relevance has been put in evidence by many authors [1,10,11,23,24] by explicit calculations and by reduction of the problem of description of nonlinear waves interaction to the resolution of systems of ordinary differential equations, as a continuation of Maslov' s theory. In other words our families of functions tend to satisfy the system modulo a remainder that tends to 0 when $\epsilon \to 0$. To construct these families we use a method which consists in solving a system of four ordinary differential equations in a Banach space whose solutions are the approximate solutions of the one pressure model. This method allows us to compute the solutions with standard convergent numerical schemes for ODEs, thus permitting comparison with existing numerical solutions of the equal pressure model obtained in scientific computing. We observe the approximate solutions we obtain agree with the results presented in [16], with a small difference in one case which diminishes in presence of the pressure correction, which can be considered as a mathematical justification of these numerical results. The system (1)–(6) is in nondivergence form, i.e. the derivatives cannot be transfered to test functions because of the terms $\alpha_i\frac{\partial p_i^{int}}{\partial x}$ in (3), (4). Therefore the study of the solutions of this system in presence of shock waves is problematic and we