Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 268–303
Journal of Differential Equations
www.elsevier.com/locate/jde

# Spreading speeds and uniqueness of traveling waves for a reaction diffusion equation with spatio-temporal delays

Zhaoquan Xu, Dongmei Xiao *,[1]

*Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China*

Received 13 March 2015; revised 6 July 2015

Available online 9 September 2015

**Abstract**

A class of reaction diffusion equation with spatio-temporal delays is systematically investigated. When the reaction function of this equation is nonlinear without monotonicity, it is shown that there exists a spreading speed $c^* > 0$ for this equation such that $c^*$ is linearly determinate and coincides with the minimal wave speed of traveling waves, and that this equation admits a unique traveling wave (up to translation) with speed $c > c^*$ and no traveling wave with $c < c^*$.


*MSC:* 35K57; 35R10; 92D25

*Keywords:* Reaction diffusion equation; Spatio-temporal delays; Spreading speeds; Traveling waves; Uniqueness

## 1. Introduction

In the study of population dynamics and disease spread, the reaction diffusion equations with spatially or temporally nonlocal delays are often used to describe biological and physical evolu-

* Corresponding author.
*E-mail addresses:* xuzhqmaths@126.com (Z. Xu), xiaodm@sjtu.edu.cn (D. Xiao).
[1] Research supported by the National Natural Science Foundation of China (No. 11371248 & No. 11431008) and the RFDP of Higher Education of China grant (No. 20130073110074).

http://dx.doi.org/10.1016/j.jde.2015.08.049

tion process, see, e.g., [1,2,8,17,18,32,33] and the references therein. In this paper, we consider a class of reaction diffusion equations with spatio-temporal delays as follows

$$w_t(t,x) = D\Delta w(t,x) + F\left(w(t,x), \int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(s,y) f\left(w(t-s,x-y)\right) dyds\right), \tag{1.1}$$

where $x \in \mathbb{R}$, $\Delta$ is the Laplacian operator on $\mathbb{R}$ and $D$ is a positive diffusion coefficient. $\mathcal{K}$ is a kernel function, which is a continuous nonnegative function with $\mathcal{K}(t,-x) = \mathcal{K}(t,x)$ and $\int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(s,y)dyds = 1$; $f$ is Lipschitz continuous on any compact interval in nonnegative real axis $\mathbb{R}_+$ with $f(0) = 0$ and $f(x) > 0$ for $x > 0$; and $F \in C^2(\mathbb{R}_+^2, \mathbb{R})$.

Some mathematical models in the literature may be described by (1.1) with appropriate choices of $F$, $\mathcal{K}$ and $f$. For example, taking $F(x,y) = -g(x) + y$, Eq. (1.1) becomes

$$w_t(t,x) = D\Delta w(t,x) - g(w(t,x)) + \int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(s,y) f(w(t-s,x-y))dyds, \tag{1.2}$$

which was studied as the evolution model of the adult population of a single species with distributed maturation delay in Gourley and So [18] and Al-Omari and Gourley [2]. Choosing $F(x,y) = -ax + b(1-x)y$ and $f(x) = x$, Eq. (1.1) can be reduced to the following vector-disease model derived by Ruan and Xiao [33]

$$w_t(t,x) = D\Delta w(t,x) - aw(t,x) + b[1-w(t,x)]\int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(s,y)w(t-s,x-y)dyds. \tag{1.3}$$

When $F(x,y) = -\tau x + \beta\tau y e^{-y}$ and $f(x) = x$, (1.1) can be reduced to the Nicholson's Blowflies equation with spatio-temporal delays

$$w_t(t,x) = D\Delta w(t,x) - \tau w(t,x) + \beta\tau[(\mathcal{K} * w)(t,x)]\exp[-(\mathcal{K} * w)(t,x)], \tag{1.4}$$

which was studied by Li et al. [26] and Lin [25], here $(\mathcal{K} * w)(t,x)$ is defined by

$$(\mathcal{K} * w)(t,x) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(s,y)w(t-s,x-y)dyds.$$

If $\mathcal{K}(t,x) = k(t)\delta(x)$ with $\delta(\cdot)$ is the Dirac delta function, (1.4) can further be reduced to the distributed time-delayed Nicholson's Blowflies equation proposed and studied by Gourley [19], and Gourley and Ruan [20]

$$w_t(t,x) = D\Delta w(t,x) - \tau w(t,x) + \beta\tau[(k * w)(t,x)]\exp[-(k * w)(t,x)], \tag{1.5}$$