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The Filippov characteristic flow for the aggregation equation with mildly singular potentials

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Abstract

Existence and uniqueness of global in time measure solution for the multidimensional aggregation equation is analyzed. Such a system can be written as a continuity equation with a velocity field computed through a self-consistent interaction potential. In Carrillo et al. (2011) [17], a well-posedness theory based on the geometric approach of gradient flows in measure metric spaces has been developed for mildly singular potentials at the origin under the basic assumption of being λ-convex. We propose here an alternative method using classical tools from PDEs. We show the existence of a characteristic flow based on Filippov's theory of discontinuous dynamical systems such that the weak measure solution is the pushforward measure with this flow. Uniqueness is obtained thanks to a contraction argument in transport distances using the λ-convexity of the potential. Moreover, we show the equivalence of this solution with the gradient flow solution. Finally, we show the convergence of a numerical scheme for general measure solutions in this framework allowing for the simulation of solutions for initial smooth densities after their first blow-up time in L^p-norms.

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1. Introduction

This paper is devoted to the so-called aggregation equation in d space dimension

$$\partial_t \rho = \operatorname{div}\big((\nabla_x W * \rho)\rho\big), \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1.1}$$

complemented with the initial condition $\rho(0,x) = \rho^{ini}$. Here, W plays the role of an interaction potential whose gradient $\nabla_x W(x-y)$ measures the relative force exerted by an infinitesimal mass localized at a point y onto an infinitesimal mass located at a point x.

This system appears in many applications in physics and population dynamics. In the framework of granular media, equation (1.1) is used to describe the large time dynamics of inhomogeneous kinetic models (see [4,16,44]). Model of crowd motion with a nonlinear dependency of the term $\nabla_x W * \rho$ is also encountered in [20,22]. In population dynamics, (1.1) provides a biologically meaningful description of aggregative phenomena. The description of the collective migration of cells by swarming leads to such non-local interaction PDEs (see e.g. [37,38, 43]). Another example is the modeling of bacterial chemotaxis. In this framework, the quantity $S = W * \rho$ is the chemoattractant concentration which is a substance emitted by bacteria allowing them to interact with each other. The dynamics can be macroscopically modeled by the Patlak–Keller–Segel system [33,39]. In the kinetic framework, the Othmer–Dunbar–Alt model is usually used, its hydrodynamic limit leading to the aggregation equation (1.1) [24,26,30]. In many of these examples, the potential W is usually mildly singular, i.e. W has a weak singularity at the origin. Due to this weak regularity, finite time blow-up of regular solutions has been observed for such systems and has gained the attention of several authors (see e.g. [34,9,5,6,17]). Finite time concentration is sometimes considered as a very simple mathematical way to mimick aggregation of individuals, as opposed to diffusion. Finally, attraction–repulsion potentials have been recently proposed as very simple models of pattern formation due to the rich structure of the set of stationary solutions, see [40,13,14,3,7] for instance.

Since finite time blow-up of regular solutions occurs, a natural framework to study the existence of global in time solutions is to work in the space of probability measures. However, several difficulties appear due to the weak regularity of the potential. In fact, the definition of the product of $\nabla W * \rho$ with ρ is a priori not well defined. This fact has already been noticed in one dimension in [30,32]. Using defect measures in a two-dimensional framework, existence of weak measure solutions for parabolic–elliptic coupled system has been obtained in [41,25]. However, uniqueness is lacking. Measure valued solutions for the 2D Keller–Segel system have been considered in [36] as limit of solutions of a regularized problem.

For the aggregation equation (1.1), a well-posedness theory for measure valued solutions has been considered using the geometrical approach of gradient flows in [17]. This technique has been extended to the case with two species in [23]. The assumptions on the potential in order to get this well-posedness theory of measure valued solutions use certain convexity of the potential that allows for mild singularity of the potential at the origin.

In this paper, we assume that the interaction potential $W : \mathbb{R}^d \to \mathbb{R}$ satisfies the following properties: