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Liouville theorems and gradient estimates for a nonlinear elliptic equation

Lin Feng Wang [1]

School of Science, Nantong University, Nantong 226007, Jiangsu, China

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Abstract

In this paper we establish gradient estimates for positive solutions to the equation

$$\triangle_f u^p = -\lambda u \tag{0.1}$$

on any smooth metric measure space whose m-Bakry–Émery curvature is bounded from below by $-(m-1)K$ with $K \geq 0$. These estimates imply Liouville theorems for (0.1). When $p \to 1$, our main theorem reduces to the gradient estimate of Wang (2010) [9]. As applications, several Harnack inequalities are obtained.

MSC: 53C21; 53C25

Keywords: Smooth metric measure space; Bakry–Émery curvature; Gradient estimate; Liouville theorem; Harnack inequality; Weighted Bochner formula

E-mail addresses: wlf711178@126.com, wlf711178@ntu.edu.cn.

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1. Introduction

The differential equation of the form

$$\partial_t u = \triangle u^p \tag{1.1}$$

for $p > 0$ is a nonlinear version of the linear heat equation. For various values of $p > 0$, it has arisen in different applications to model diffusive phenomena. For $p = 1$ it is the well-known linear heat equation. When $p > 1$, equation (1.1) is known as the porous medium equation, which models the flow of gas through porous medium, ground water filtration, heat radiation in plasmas, etc. [7,11]. In the case that $0 < p < 1$, equation (1.1) becomes the so-called fast diffusion equation.

When $p > 1$, Aronson–Bénilan and Li–Yau type estimates for positive solutions to (1.1) on complete manifolds with nonnegative Ricci curvature were firstly proved by Vázquez [11]. Under the weaker assumption that the Ricci curvature of M is bounded from below by $-K$ for some $K \geq 0$, Lu, Ni, Vázquez and Villani [7] derived more general Aronson–Bénilan and Li–Yau type gradient estimates for (1.1) when $p > 0$. See also the related work in [4]. More recently, Cao and Zhu [2] studied the porous medium equation (1.1) ($p > 1$) coupled with the Ricci flow on complete manifolds with bounded nonnegative curvature operator.

Inspired by (1.1), we study the following nonlinear elliptic equation

$$\triangle_f u^p = -\lambda u \tag{1.2}$$

on smooth metric measure spaces with Bakry–Émery curvature bounded from below, where $p > 0$ and λ are constant. Recall that a smooth metric measure space is a triple $(M, \mathrm{g}, \mathrm{d}\mu)$, where $\mathrm{d}\mu = e^{-f(x)}\mathrm{d}x$ is the weighted measure for some smooth potential function f. The weighted (shifting) Laplacian is defined by

$$\triangle_f = \triangle - \nabla f \cdot \nabla.$$

We always use the m-Bakry–Émery curvature

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\ f - \frac{1}{m-n}\mathrm{d}f \otimes \mathrm{d}f$$

to replace the Ricci curvature when studying the weighted measure, where $m > n$. The ∞-Bakry–Émery curvature is

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\ f.$$

It is well-known that $\triangle_f$ relates to the ∞-Bakry–Émery curvature via the following weighted Bochner formula [6,9,10]

$$\frac{1}{2}\triangle_f |\nabla u|^2 = |\nabla^2 u|^2 + \nabla u \cdot \nabla \triangle_f u + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{1.3}$$

Bakry and Qian [1] derived the weighted Laplacian comparison theorem and the growth estimate of the weighted volume when the m-Bakry–Émery curvature is bounded from below.