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Singular levels and topological invariants of Morse Bott integrable systems on surfaces

J. Martínez-Alfaro [a,*], I.S. Meza-Sarmiento [b], R.D.S. Oliveira [b]

[a] *Department de Matemática Aplicada, Universitat de València, Burjassot, Spain*
[b] *Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13566-590 – São Carlos, SP, Brazil*

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Abstract

We classify up to homeomorphisms closed curves and eights of saddle points on orientable closed surfaces. This classification is applied to Morse Bott foliations and Morse Bott integrable systems allowing us to define a complete invariant. We state also a realization Theorem based in two transformations and one generator (the foliation of the sphere with two centers).

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1. Introduction

The research on topological invariants of flows $X(\Sigma)$ and foliations $\mathcal{F}(\Sigma)$ on surfaces Σ has a long history. Some basic references are: [1–4] and the references cited in these papers. An introduction to the subject can be also found in the book [5] and the relation with C^*-algebras is

* Corresponding author.
E-mail addresses: martinja@uv.es (J. Martínez-Alfaro), isofia@icmc.usp.br (I.S. Meza-Sarmiento), regilene@icmc.usp.br (R.D.S. Oliveira).

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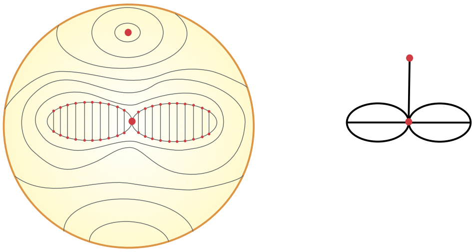

Fig. 1. System with one center and its orbit space.

analyzed in [6] and [7]. At present a lot of efforts are devoted to singular foliations and integrable flows on manifolds of larger dimensions, see for instance [8–13]. Nevertheless, as we try to show in this paper, the two dimensional case is surprisingly incomplete.

Recall that two systems $X_1(\Sigma)$ and $X_2(\Sigma)$ are *topologically equivalent* if there exists a homeomorphism $h : \Sigma \to \Sigma$ that sends orbits of $X_1(\Sigma)$ into orbits of $X_2(\Sigma)$ preserving the sense of the orbits. To describe the equivalence classes it is useful to define a set of topological invariants.

We try to sketch the usual method of construction of invariants. The strategy lies in several initial reductions of the system. The first one consists of the construction of the space of orbits Σ/X (or $\Sigma/\mathcal{F}$, space of leaves in the case of a foliation), that is to say: two points belong to the same class if and only if they lie on the same orbit. If some orbits can be related, for instance their union is a one dimensional manifold, a reduced space of orbits, $(\Sigma/X, \sim)$ could be defined. Usually $(\Sigma/X, \sim)$ is a singular foliation. In a second step, a finite number of open regions in $(\Sigma/X, \sim)$ are defined. In each region the orbits has a homogeneous kind of behavior. They have the same asymptotic properties or define a parallel flow. Some kind of graph Γ whose vertices or edges are the homogeneous regions encodes the relation between these regions. To recover the structure of the initial system additional information is added to the graph, basically specifications of local flows or order. The graph and the additional information define the invariant.

If the additional information is not enough to reconstruct the flow one gets an invariant that is not complete. A consideration usually overlooked is that if two systems are equivalent the topological type of two corresponding invariant sets must be the same. In Example 2, Figs. 6, 7 we have two foliations with isomorphic space of leaves, but the foliations are not topologically equivalent because some singular leaves are not topologically equivalent.

Following the described guidelines, in Section 3 we classify the basic leaves, closed curves and saddles with their separatrices according to their topological type. In this case, the topological equivalence is similar to the equivalence used to define knots on S^3. The particular case of closed curves in the torus is analyzed in [14], page 25. This classification can be applied to almost all flows and foliations on a surface and in fact is an independent part of the paper. This classification closes a remarkable gap in the study of two-dimensional systems.

If Γ determines the surface Σ, for instance by the number of cycles, the topological type of the basic leaves can be implicitly included in the invariant. This is particularly true for Morse Bott foliations (see Section 2) and Morse Bott integrable systems. But this is not always the case:

Example 1. The orbit space Σ/X of the system defined on the 2-sphere and represented in Fig. 1 contains four cycles. The sphere is not contractible to Σ/X.