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Smooth solutions of the one-dimensional compressible Euler equation with gravity

Cheng-Hsiung Hsu [a,*,1], Song-Sun Lin [b,2], Chi-Ru Yang [c,3]

[a] *Department of Mathematics, National Central University, Chungli 32001, Taiwan*
[b] *Department of Applied Mathematics, National Chiao-Tung University, Hsinchu 30050, Taiwan*
[c] *Mathematics Division, National Center for Theoretical Sciences, Hsinchu 30043, Taiwan*

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Dedication to the retirement of Professor Tetu Makino

Abstract

We study one-dimensional motions of polytropic gas governed by the compressible Euler equations. The problem on the half space under a constant gravity gives an equilibrium which has free boundary touching the vacuum and the linearized approximation at this equilibrium gives time periodic solutions. But it is difficult to justify the existence of long-time true solutions for which this time periodic solution is the first approximation. The situation is in contrast to the problem of free motions without gravity. The reason is that the usual iteration method for quasilinear hyperbolic problem cannot be used because of the loss of regularities which causes from the touch with the vacuum. Due to this reason, we try to find a family of solutions expanded by a small parameter and apply the Nash–Moser Theorem to justify this expansion. Note that the application of Nash–Moser Theorem is necessary for the sake of conquest of the trouble with loss of regularities, and the justification of the applicability requires a very delicate analysis of the problem.

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* Corresponding author.
E-mail addresses: chhsu@math.ncu.edu.tw (C.-H. Hsu), sslin@math.nctu.edu.tw (S.-S. Lin), cryang@math.cts.nthu.edu.tw (C.-R. Yang).

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1. Introduction

The aim of this paper is to study one-dimensional motions of polytropic gas governed by the compressible Euler equations

$$\rho_t + (\rho u)_x = 0, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + P)_x = -g\rho, \tag{2}$$

for $t, x \geq 0$ subject to the boundary condition

$$\rho u|_{x=0} = 0. \tag{3}$$

Here ρ, u, P and $g > 0$ are density, velocity, pressure and gravitational acceleration constant respectively. Equations (1) $\sim$ (3) describe the atmosphere on the flat earth $\{x \leq 0\}$ moving in one direction under the constant gravitational force downward. For more descriptions about the physical background of the above system, we refer the readers to the book of Chandrasekhar [1].

In this work, we assume that $P = P(\rho) := A\rho^\gamma$ for some constants A, γ such that $A > 0$, $1 < \gamma \leq 2$. Then equilibria of (1) and (2) are of the form

$$\bar{\rho}(x) = \begin{cases} A_1(x_+ - x)^{\frac{1}{\gamma-1}}, & \text{if } 0 \leq x \leq x_+, \\ 0, & \text{if } x_+ < x, \end{cases} \tag{4}$$

where $A_1 := ((\gamma-1)g/\gamma A)^{1/(\gamma-1)}$ and x_+ is an arbitrary positive value which represents the stratospheric depth.

Without loss of generality, we may assume $x_+ = 1$, $A_1 = 1$ and $A = 1/\gamma$, which can be seen easily by scale transformations of the variables. Since the interface with the vacuum would vary with the time, it is convenient to transform the equations (1) and (2) into the Lagrangian form. More precisely, we introduce the variable

$$m := \int_0^x \rho dx$$

as the independent variable instead of x, then equations (1) and (2) can be transformed into the following second order equation:

$$x_{tt} + P_m = -g, \tag{5}$$

where $P = \gamma^{-1}(x_m)^{-\gamma}$. Let us fix an equilibrium

$$x = \bar{x}(m) := 1 - A_2(m_+ - m)^{\frac{\gamma-1}{\gamma}}, \quad 0 \leq m \leq m_+, \tag{6}$$