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Solutions of fractional Laplacian equations and their Morse indices

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Abstract

In this paper, we study the solutions of the following fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c. \end{cases}$$

Under some assumptions on the nonlinearity, we will show that the boundedness of solutions is equivalent to the finiteness of their Morse indices. We extend the result of Bahri and Lions (1992) [1], from the Laplacian equations to the fractional Laplacian equations.

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1. Introduction

In this paper, we study the properties of solutions for the following fractional Laplacian equations

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$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $s \in (0,1)$ is a fixed constant and $(-\Delta)^s$ is the fractional Laplacian operator. In general, the fractional Laplacian operator is not defined according to the derivatives, it is defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, dy, \quad x \in \Omega. \tag{1.2}$$

It is easy to see from the definition that the fractional Laplacian is a nonlocal operator. So the homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^N \setminus \Omega$ rather than simply on $\partial\Omega$.

The fractional Laplacian equations appear in many areas such as quantum mechanics and finance. In quantum mechanics, it is well known that the path integral over Brownian trajectories leads to the classical Schrödinger equation. However, it was shown by Montroll in [16] that for a free quantum mechanical particle the chain condition for the kernel can be solved and leads to a kernel that is the quantum analog of the Levy transition probability. Hence, the path integral over Levy-line quantum mechanical path is important in quantum mechanics. Moreover, it was proved in [15] that the path integral over Levy trajectories leads to the fractional Schrödinger equation. Fractional Laplacian equation is also widely used in finance, especially on modeling the price of European and American options. In the traditional Black–Scholes formula, the price of the underline assets is assumed to be a Gaussian process. This assumption is too severe since the stock is only exchanged in the market days. So the price of stock can't be a Gaussian process. A natural extension of this formula is that the price of the underline assets is not Gaussian, for example, the Levy process or a jump-diffusion process. Then the price of the European call option satisfies an equation governed by a fractional Laplacian operator.

On the other hand, from the mathematicians' point of view, much attention has been paid to the study of fractional Laplacian equations during the past few years. One of the main difficulties lies in that the fractional Laplacian operator is a nonlocal operator. The greatest achievement in overcoming this difficulty is the extension theorem by L. Caffarelli and L. Silvestre in [8]. After some extension, the authors transformed the nonlocal problem into a local problem. After the work [8], a great deal of progress has been made to the fractional Laplacian equations. For example, X. Cabre and Y. Sire studied the regularity result, maximum principle, existence result, uniqueness result and other properties in [4,5]. Other results can be found in [2,3,6,7,14,20–24]. Also, there are some papers which deal with the fractional Laplacian operator directly from its definition, we refer the readers to [9,19] and etc.

In this paper, we mainly concern with the prior bound and the Morse indices of solutions for problem (1.1). In a famous paper [1], the authors studied the Laplacian equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

The authors proved that the solutions of problem (1.3) are bounded if and only if their Morse indices are finite. The proof was based on the blow up method and the Liouville type theorem with finite Morse index. Later, A. Harrabi, S. Rebhi and S. Selmi extended their results to more general nonlinearities in [11,12]. Recently, A. Harrabi, M. Ahmedou, S. Rebhi and A. Selmi