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On the lifespan of solutions to nonlinear Cauchy problems with small analytic data

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Abstract

The paper studies the lifespan of solutions to Cauchy problems for nonlinear analytic partial differential equations with small analytic data. It is proved that the lifespan of the solution becomes longer as the initial data become smaller. The dependence of the lifespan on the smallness of the data can be sharply described by the property of the equation.

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1. Introduction

The effect of small initial data on the solutions of Cauchy problems has been extensively studied by different authors. Most of the research focused on the interrelated notions of lifespan, finite-time blowup, and global solvability. In this paper, we consider the lifespan of solutions to general nonlinear partial differential equations from the standpoint of the Cauchy–Kowalevsky theorem.

Denote by $(t, x) = (t, x_1, \dots, x_n)$ the variables in $\mathbb{R} \times \mathbb{R}^n$. Set $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$, and $I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n ; \ j + |\alpha| \le m, \ j < m\}$ (where $m \in \mathbb{N}^*$, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$). Let $N = \# I_m$ and denote by $X = \{X_{j,\alpha}\}_{(j,\alpha)\in I_m}$ the variables in $\mathbb{C}^N$.

Let Ω be an open subset of $\mathbb{R}^n$, $\mathscr{U}$ be a convex open neighborhood of the origin of $\mathbb{C}^N$, and $F(t, X)$ be a function on $\mathbb{R}_t \times \mathscr{U}$. In this paper, we consider the Cauchy problem

$$\begin{cases} \partial_t^m u = F\big(t, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big), \\ \partial_t^i u(0, x) = \phi_i(x), \quad i = 0, 1, \dots, m-1 \end{cases} \tag{1.1}$$

under the following assumptions:

(A_1) $F(t, X)$ is continuous and bounded on $\mathbb{R}_t \times \mathscr{U}$, and holomorphic on $\mathscr{U}$ for any fixed $t \in \mathbb{R}$;
(A_2) $F(t, 0) \equiv 0$ for any $t \in \mathbb{R}$;
(A_3) $\phi_i(x) \in \mathcal{A}(\Omega)$ $(0 \le i \le m-1)$, where $\mathcal{A}(\Omega)$ is defined below.

Definition 1.1. (1) A C^∞-function $f(x)$ on Ω is said to be uniformly analytic on Ω if there are $C > 0$ and $h > 0$ such that

$$\sup_{x\in\Omega} |\partial_x^\alpha f(x)| \le C h^{|\alpha|} |\alpha|!$$

for any $\alpha \in \mathbb{N}^n$. We define the function space $\mathcal{A}(\Omega)$ to be the totality of uniformly analytic functions on Ω.

(2) Let $T > 0$ and $k \in \mathbb{N}$. A function $u(t, x)$ is said to belong to $C^k((-T, T), \mathcal{A}(\Omega))$ if the following hold for any $j \in \{0, 1, \dots, k\}$ and $\alpha \in \mathbb{N}^n$:

(i) $\partial_t^j \partial_x^\alpha u(t, x) \in C^0((-T, T) \times \Omega)$,
(ii) for any $0 < T_1 < T$, there are $C_1 > 0$ and $h_1 > 0$ such that

$$\sup_{(-T_1,T_1)\times\Omega} |\partial_t^j \partial_x^\alpha u(t, x)| \le C_1 {h_1}^{|\alpha|} |\alpha|!.$$

It is well known that (1.1) has a unique local solution $u(t, x) \in C^m((-\delta, \delta), \mathcal{A}(\Omega))$ for some $\delta > 0$. This fact was first proved by Nagumo [8], and alternative proofs were given by Nirenberg [9], Nishida [10] and Walter [12]. Then the lifespan of the solution is defined as the supremum of all such δ.

D'Ancona and Spagnolo [1,2] investigated the lifespan of the solution to (1.1) in the analytic category for nonlinear hyperbolic equations or systems with initial data $\phi_i(x) = \varepsilon u_i(x)$ (where $\varepsilon > 0$ is a parameter) and they proved that the lifespan T_ε tends to ∞ as $\varepsilon \to 0$. Moreover, in