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Riemann solutions without an intermediate constant state for a system of two conservation laws

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Abstract

We present a class of systems consisting of two conservation laws in one spatial dimension that share an intriguing property: they admit structurally stable Riemann solutions without the standard constant state. This striking phenomenon emerges in sharp contrast to what is known for strictly hyperbolic systems of conservation laws, in which the existence of constant states is necessary for the structural stability of Riemann solutions. We prove that, together, coincidence of characteristic speeds and a certain amount of genuine nonlinearity are sufficient to trigger the aforementioned phenomenon. The proof revolves about the presence of a singular point in the coincidence set that organizes the construction of our Riemann solutions.

Keywords: Hyperbolic conservation laws; Riemann problem; Structural stability

1. Introduction

Partial differential equations are a common language for describing processes taking place in a *continuum*. Notorious among such class of equations are those of the hyperbolic type, which model the (finite speed) transport of information. Information may refer to density, momentum

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or the dilution of a chemical component, to cite a few common examples. We are interested in hyperbolic systems of conservation laws in one spatial dimension. In such systems, hyperbolicity manifests itself through the characteristic speeds of the system: real quantities intimately related to the motion. Albeit actively developed in the case of strict hyperbolicity (distinct characteristic speeds), the theory is almost nonexistent in the case of coinciding characteristic speeds we consider.

The framework we use is the Riemann problem. For hyperbolic conservation laws in one space dimension, the Riemann problem is an initial value problem with initial data consisting of two constant states separated by a jump discontinuity at the origin. It is well known that under the hypotheses of strict hyperbolicity and genuine nonlinearity, see Lax [8], the Riemann problem is locally solvable in the class of constant states and elementary wave fans: rarefaction waves and shock discontinuities. This result was extended by Liu [9,10], who was able to solve the Riemann problem for systems where genuine nonlinearity does not hold globally. There, rarefaction and shock waves could be concatenated in groups of elementary waves moving together as a single entity, the *wave-groups*. As in the genuinely nonlinear case, however, the hypothesis of strict hyperbolicity ultimately leads the wave-groups in the Riemann solutions to move apart from each other, leaving a constant state in the sector between two consecutive groups. More than that, the Riemann solutions are also stable, in the sense that they preserve their structure upon variation of initial (Riemann) data. This structural stability was the objective of Furtado's unpublished thesis [3] and Schecter et al. [13].

In the unpublished thesis [14], a full Riemann solution was exhibited for a system of three hyperbolic conservation laws that arises in petroleum engineering, modeling compositional thermal two-phase flow in a porous medium. That model allows the characteristic speeds to coincide on a curve in state space. The nonlinear resonance in that system produces a curious feature that claims more attention: it exhibits a structurally stable Riemann solution without constant states [15]. Such discovery emerges in sharp contrast to what is known for the strictly hyperbolic case [3] and [13], in which the existence of constant states is necessary for the structural stability of Riemann solutions. Yet more striking is the fact that, at least to our knowledge, little evidence has appeared prior to [14] that a different behavior should be expected when a mild loss of strict hyperbolicity occurs. The only hint was given by systems of equations that possess an umbilic point [5]; however the corresponding Riemann solutions require two constant states.

Our main objective here is to shed light onto Riemann solutions without constant states. To this end, we coined a system of two conservation laws that reproduces, in the simplest way we could find, the wave patterns and interactions found in [14,15] near a certain organizing center. For such system one is able to prove, in an intelligible way, the following:

Theorem 1.1. *There are open sets U, V in state space such that there exists a structurally stable Riemann solution for any pair $(w_L, w_R) \in U \times V$ of initial data. Such Riemann solutions possess no intermediate constant state.*

This manuscript is organized as follows. In Section 2 we provide basic facts about Riemann solutions and set the framework that we will use herein. Sections 3 and 4 are devoted to the construction of elementary wave-curves: rarefaction and shock curves, respectively. The topology of such curves is deeply related to a certain singular point in state space. There, we lay the raw facts that will be needed to construct the wave-curves; the main tool we use to build the Riemann solutions in Theorem 1.1. In Section 5 we construct an *extension* map that will be used to glue wave-curves, disallowing constant states. To this end an open double-contact locus is needed;