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On codimension one foliations with prescribed cuspidal separatrix

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Abstract

In this paper, we will construct a pre-normal form for germs of codimension one holomorphic foliation having a particular type of separatrix, of cuspidal type. As an application, we will explain how this form could be used in order to study the analytic classification of the singularities via the projective holonomy, in the generalized surface case. We will also give an application to the analytic classification of singularities, and a sufficient condition, in the quasi-homogeneous, three-dimensional case, for these foliations to be of generalized surface type.

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1. Introduction and statement of the results

The objective of this paper is to find a (pre)-normal form for holomorphic differential 1-forms defining local holomorphic foliations in $(\mathbb{C}^{n+1}, \mathbf{0})$, with a fixed separatrix of a certain particular type, that we will describe later, and with a condition on the multiplicity at the origin. As an application, we will use this pre-normal form to describe a Hopf fibration, constructed after reduction of the singularities, transverse to the foliation away from the separatrices, which turns out to be a useful tool to study the analytic classification of the singularities of the foliations via the projective holonomy. As a side result, a condition in dimension three for some of these forms, that will be called cuspidal, in order that they are of generalized surface type, is obtained.

The search of normal forms for germs of holomorphic vector fields, or of holomorphic 1-forms, is carried out in several papers, either treating directly this problem, or as a need for treating another problems, such as the formal, analytic or topological classification of these objects. It turns out to be a useful tool in the study of the singularities of these objects. Let us briefly define the main concepts that we will use throughout the paper, and give an account of some of the main achievements in dimensions two and three, focusing in the case of codimension one foliations.

A germ of codimension one holomorphic foliation $\mathcal{F}$ in $(\mathbb{C}^n, \mathbf{0})$ is defined by a holomorphic 1-form ω, satisfying the Frobenius integrability condition $\omega \wedge d\omega = 0$, and such that its coefficients have no common factors. If $\omega = \sum_{i=1}^n a_i(\mathbf{x})\, dx_i$, the *singular set* of ω, $\mathrm{Sing}(\omega)$, is the germ of analytic set defined by the zeros of the ideal $(a_i(\mathbf{x})_{1 \leqslant i \leqslant n})$. Previous condition implies that $\mathrm{Sing}(\omega)$ has codimension at least two.

We will also consider throughout the paper meromorphic integrable 1-forms: these are integrable 1-forms $\omega = \sum_{i=1}^n a_i(\mathbf{x})\, dx_i$, where $a_i(\mathbf{x})$ are germs of meromorphic functions. Consider an $(n-1)$-dimensional germ of analytic set, S, defined by an equation $(f = 0)$, with $f \in \mathcal{O}$ reduced (here and throughout the paper, $\mathcal{O} = \mathcal{O}_n$ will denote the ring $\mathbb{C}\{\mathbf{x}\}$ of convergent power series in n variables, where n is usually omitted as there is no risk of confusion).

A meromorphic 1-form is called *logarithmic* along S if $f\omega$ and $f\, d\omega$ are holomorphic forms, or equivalently if $f\omega$ and $df \wedge \omega$ are holomorphic. The following result follows:

Proposition 1.1. *(See [15,31].) Let ω be a holomorphic 1-form, and f as before. The following conditions are equivalent*:

(1) *There exists a holomorphic 2-form η such that $\omega \wedge df = f\eta$.*
(2) *ω/f is logarithmic along S.*
(3) *There exist $g, h \in \mathcal{O}$, and a holomorphic 1-form α, such that $g\omega + h\, df = f\alpha$, and moreover, g, f have no common factors.*

The equivalence between (1) and (3) can be read in [15], assuming irreducibility of f. The result is stated in [31] in a more general context (q-forms), without the irreducibility assumption in the statement, but the proof provided there is only valid in the irreducible case. Nevertheless, a modification of that proof is enough to establish the result in the non-irreducible case.

If $\mathcal{F}$ is a holomorphic foliation defined by a 1-form ω satisfying the conditions of the previous theorem, we will say that S (or $f = 0$) is a separatrix for $\mathcal{F}$. Let us observe that, if $\Omega^1(\log S)$ represents the set of meromorphic 1-forms having f as separatrix (i.e. logarithmic along S), $\Omega^1(\log S)$ has a structure of $\mathcal{O}$-module. Under some condition, this $\mathcal{O}$-module is free. In fact, K. Saito proves in [31] the following result: