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Flow reconstruction and invariant tori in the spatial three-body problem ☆

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Abstract

We deal with the spatial three-body problem in the various regimes where the Hamiltonian is split as the sum of two Keplerian systems plus a small perturbation. By averaging over the mean anomalies, truncating higher-order terms and using singular reduction theory we get a one-degree-of-freedom Hamiltonian system. Departing from the analysis performed in [39] concerning the relative equilibria of this reduced system, we carry out the reconstruction of the KAM tori surrounding the motions associated to each elliptic equilibrium. The existence of five-dimensional KAM tori for the spatial three-body problem is established. These tori surround various types of motions, from circular to near rectilinear, passing through coplanar or perpendicular.

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1. Introduction

The aim of this paper is to classify different types of motions in the spatial three-body problem. Starting from the analysis of the relative equilibria of the one-degree-of-freedom reduced system obtained in [39] we reconstruct the flow of the full problem. We show that the different motions of the three bodies have to be studied in the adequate reduced (orbit) spaces accordingly to their level of degeneracy and that specific variables should be designed for achieving this study. Our analysis is valid in the different regimes where the system can be decomposed as the sum of an integrable part of Keplerian type plus a small regular perturbation.

The topic has attracted the attention of great mathematicians since Laplace times, but with the advent of KAM theory, the problem achieved a renewed interest. We could point out the pioneering work by Arnold [3], who tackles the existence of $(3N-4)$-dimensional Lagrangian Diophantine tori for the N-body problem. He considers the case where all bodies move in near-circular–coplanar orbits and gives a rigorous complete proof for the planar three-body problem and some indications for the generalisation to the spatial case and to N bodies.

Concerning the near-circular–coplanar motions in the planetary case, where one body dominates the system and the others are small, Robutel [42] extends Arnold's result to the spatial planetary three-body problem. The existence of quasi-periodic motions for almost all values of the ratio of the semi-major axis and almost all values of the mutual inclination up to about one degree is proved. Biasco, Chierchia and Valdinoci [5] deal with the case of lower-dimensional tori, proving the existence of two-dimensional KAM tori in the spatial three-body problem. Féjoz [22] (following Herman) gives a complete proof of 'Arnold's Theorem' on the planetary N-body problem, establishing the existence of a positive measure set of smooth Lagrangian invariant tori. The analytic version of the invariant tori is due to Chierchia and Pusateri [10]. Another direct proof of Arnold's Theorem as well as the existence of elliptic lower dimensional tori is carried out by Chierchia and Pinzari [8,9].

Only a few results are known outside the near-circular–coplanar regime. Jefferys and Moser [27] prove the existence of two- and three-dimensional invariant tori for the spatial three-body problem. The three bodies move around their centre of mass in quasi-periodic orbits that are nearly circular and inclined. They find these motions in two situations, the planetary case and the lunar case, where the mass ratios are arbitrary but the ratio of the two semimajor axes is small. In the planar case Lieberman [31] analyses the relative equilibria, together with their stable character and bifurcations. More recently Féjoz [20,21] determined the quasi-periodic motions related to the relative equilibria of elliptic and hyperbolic character obtained after reducing out the symmetries of the problem. These solutions belong to what he calls the perturbing region, where the Hamiltonian splits as the sum of two Keplerian systems plus a smaller perturbation. By using singular reduction and a combination of analytical and numerical studies, Cordani [11] confirms Féjoz's conjecture on the number of relative equilibria of the two-dimensional reduced system. Recently Zhao in his thesis [48] (see also [49,51]) used Herman and Féjoz's ideas on special KAM theorems valid for degenerate cases to obtain a large variety of quasi-periodic solutions, including near-circular–coplanar and almost-collision orbits in the lunar case of the spatial three-body problem.

The contents of the paper are part of the second author's PhD thesis [44].

In [39] we deal with the reduction of the spatial three-body problem and the analysis of the simplest reduced Hamiltonian. Specifically, we introduce Jacobi coordinates to reduce the translation symmetry. Then, we attach the reference frame to the centre of mass of the system, in this way the resulting Hamiltonian has six degrees of freedom. In order to apply perturbation theory