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Nonlinear parabolic flows with dynamic flux on the boundary

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Abstract

A nonlinear divergence parabolic equation with dynamic boundary conditions of Wentzell type is studied. The existence and uniqueness of a strong solution is obtained as the limit of a finite difference scheme, in the time dependent case and via a semigroup approach in the time-invariant case.

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1. Introduction

This paper deals with the well-posedness of a nonlinear parabolic equation posed in a bounded regular domain Ω (of class C^2, for instance) of $\mathbb{R}^N$, $N \geq 1$, coupled with a dynamic boundary condition of reaction–diffusion type. More exactly, we study the problem

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$$y_t - \nabla \cdot \beta(t,x,\nabla y) \ni f, \quad \text{in } Q := (0,T) \times \Omega, \tag{1.1}$$

$$\beta(t,x,\nabla y) \cdot \nu + y_t \ni g, \quad \text{on } \Sigma := (0,T) \times \Gamma, \tag{1.2}$$

$$y(0) = y_0, \quad \text{in } \Omega, \tag{1.3}$$

where $t \in (0,T)$, $T < \infty$, $x \in \Omega$, Γ is the boundary of Ω, ν is the outward normal to Γ, $y_t = \frac{\partial y}{\partial t}$ and $\nabla y = \{\frac{\partial y}{\partial x_i}\}_{i=1,\dots,N}$.

In relation to various cases studied in this paper, certain combinations of the following hypotheses will be used:

(H_1) For each $(t,x) \in \overline{Q}$, $\beta : \overline{Q} \times \mathbb{R}^N \to \mathbb{R}^N$ is a maximal monotone graph with respect to r on $\mathbb{R}^N \times \mathbb{R}^N$, and it is derived from a potential $j(t,x,r)$. The function j is continuous on $\overline{Q} \times \mathbb{R}^N$, and for each $(t,x) \in \overline{Q}$, it is convex with respect to r. We denote

$$\partial j(t,x,r) = \beta(t,x,r), \quad \text{for any } r \in \mathbb{R}^N,\ t \in [0,T],\ x \in \overline{\Omega}, \tag{1.4}$$

where $\partial j(t,x,\cdot)$ denotes the subdifferential of $j(t,x,\cdot)$, that is

$$\partial j(t,x,r) = \big\{ w \in \mathbb{R}^N;\ j(t,x,r) - j(t,x,\bar{r}) \le w(r - \bar{r}),\ \forall \bar{r} \in \mathbb{R}^N \big\}. \tag{1.5}$$

Moreover, there is

$$\xi_0 \in C\big(\overline{Q}; \mathbb{R}^N\big) \text{ with } \nabla \cdot \xi_0 \in L^2(\Omega), \quad \text{such that } \xi_0(t,x) \in \beta(t,x,0),\ \forall (t,x) \in \overline{Q}. \tag{1.6}$$

(H_2) (*Strong coercivity hypothesis*) There exist $C_i, C_i^0 \in \mathbb{R}$, $C_1, C_2 > 0$, such that

$$C_1 |r|_N^p + C_1^0 \le j(t,x,r) \le C_2 |r|_N^p + C_2^0, \quad \forall (t,x) \in \overline{Q},\ \text{for } 1 < p < \infty. \tag{1.7}$$

(All over, $|\cdot|_N$ will denote the Euclidean norm in $\mathbb{R}^N$.)

(H_3) (*Weak coercivity hypothesis*) The functions j and j^* satisfy

$$\lim_{|r|_N \to \infty} \frac{j(t,x,r)}{|r|_N} = +\infty, \quad \text{uniformly with respect to } t, x, \tag{1.8}$$

$$\lim_{|\omega|_N \to \infty} \frac{j^*(t,x,\omega)}{|\omega|_N} = +\infty, \quad \text{uniformly with respect to } t, x, \tag{1.9}$$

where $j^* : \overline{Q} \times \mathbb{R}^N \to \mathbb{R}$ is the conjugate of j, defined by

$$j^*(t,x,\omega) = \sup_{r \in \mathbb{R}^N} \big(\omega \cdot r - j(t,x,r) \big), \quad \text{for all } \omega \in \mathbb{R}^N,\ \forall (t,x) \in \overline{Q}. \tag{1.10}$$

We note that (1.8) and (1.9) are equivalent to

$$\sup\big\{ |r|_N;\ r \in \beta^{-1}(t,x,\omega),\ |\omega|_N \le M \big\} \le W_M, \tag{1.11}$$

$$\sup\big\{ |\omega|_N;\ \omega \in \beta(t,x,r),\ |r|_N \le M \big\} \le Y_M, \tag{1.12}$$

respectively, where M, W_M, Y_M are positive constants.