Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 259 (2015) 1542–1559
Journal of Differential Equations
www.elsevier.com/locate/jde

Wei–Norman equations for classical groups

Szymon Charzyński [a,*], Marek Kuś [b]

[a] *Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warszawa, Poland*

[b] *Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warszawa, Poland*

Received 16 December 2013; revised 3 March 2015

Available online 13 March 2015

Abstract

We show that the nonlinear autonomous Wei–Norman equations, expressing the solution of a linear system of non-autonomous equations on a Lie algebra, can be reduced to the hierarchy of matrix Riccati equations in the case of all classical simple Lie algebras. The result generalizes our previous one concerning the complex Lie algebra of the special linear group. We show that it cannot be extended to all simple Lie algebras, in particular to the exceptional G_2 algebra.

Keywords: Lie group; Lie algebra; Riccati; ODE; Lie system; Root system

1. Introduction

For a non-autonomous system of N linear differential equations

$$\frac{d}{dt}\mathbf{x}(t) = M(t)\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t)$ is an N dimensional vector and the $N \times N$ coefficient matrix is time-dependent it is, in general, difficult to find a solution in a finite form (not invoking a series expansion). A possible non-commutativity of the coefficient matrices $M(t)$ and $M(t')$ calculated in different times t, t'

* Corresponding author.
E-mail addresses: szycha@fuw.edu.pl (S. Charzyński), marek.kus@cft.edu.pl (M. Kuś).

http://dx.doi.org/10.1016/j.jde.2015.03.002

prevents, namely, an explicit calculation of the ordered time product of exponentials of the time dependent matrix variable.

Wei and Norman [1,2] proposed a method for solving such a system of equations by transforming it to an autonomous, albeit nonlinear system using Lie-algebras techniques. Roughly speaking, the method consists of writing $M(t)$ as a linear combination with time dependent coefficients of generators X_i of a Lie algebra, $M(t) = \sum_i a_i(t)X_i$, and looking for the solution in terms of a product of exponentials of the generators (i.e. elements of the corresponding Lie group), $\mathbf{x}(t) = \prod_i \exp(u_i(t)X_i)\mathbf{x}(0)$. The resulting system of nonlinear equations derived from the original, non-autonomous, linear one (1) connects the unknown functions $u_i(t)$ with the coefficients $a_k(t)$.

In [3] we have shown that using the Wei–Norman technique [1,2] in the unitary case, i.e. when the solution of the linear system is given by a unitary evolution operator, the nonlinear system by an appropriate choice of ordering can be reduced to a hierarchy of matrix Riccati equations. To this end we have considered a general linear non-autonomous dynamical system on the special linear group $SL(N+1,\mathbb{C})$. The algebra $\mathfrak{sl}(N+1,\mathbb{C})$ is one of the classical Lie algebras, of the type denoted by A_N. In the present paper we generalize the method developed in [3] to all classical Lie algebras, namely the algebras of type A_N, B_N, C_N and D_N. In particular we show that in all classical cases the resulting nonlinear system can be always reduced to a hierarchy of matrix Riccati equations if we chose an appropriate ordered basis of the underlying Lie-algebra. Thus, we expand the applicability of the Wei–Norman method from the unitary and special linear groups to orthogonal and symplectic groups.

To exhibit the Riccati structure of the Wei–Norman equations in the case of A_N algebras considered in [3] we proved several facts concerning their structural properties, among them two crucial ones. The first establishing a decomposition of the $\mathfrak{sl}(N+1,\mathbb{C})$ Lie algebra into a semidirect sum of Abelian subalgebras and the second concerning the nilpotency of the adjoint endomorphism. Here we show that these two observations remain valid for all classical simple Lie algebras. Interestingly they cannot be extended to all simple Lie algebras – we show this in the case of an exceptional Lie algebra G_2.

2. The Wei–Norman method

We briefly recall our analysis of the Wei–Norman method presented in greater detail in [3]. Let G be an n-dimensional Lie group and $\mathfrak{g}$ – its Lie algebra. We assume in the following that $\mathfrak{g}$ is a simple complex classical Lie algebra. Let also $\mathbb{R} \ni t \mapsto M(t) \in \mathfrak{g}$ be a curve in $\mathfrak{g}$ and $K(t)$ – a curve in G given by the differential equation

$$\frac{d}{dt}K(t) = M(t)K(t), \qquad K(0) = I. \tag{2}$$

In $\mathfrak{g}$ we choose some basis X_k, $k = 1, \dots, n$ in which $M(t)$ takes the form

$$M(t) = \sum_{k=1}^{n} a_k(t)X_k. \tag{3}$$

We look for the solution $K(t)$ in the form

$$K(t) = \prod_{k=1}^{n} \exp\big(u_k(t)X_k\big), \tag{4}$$