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Modeling and analysis of a phase field system for damage and phase separation processes in solids ☆

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Abstract

In this work, we analytically investigate a multi-component system for describing phase separation and damage processes in solids. The model consists of a parabolic diffusion equation of fourth order for the concentration coupled with an elliptic system with material dependent coefficients for the strain tensor and a doubly nonlinear differential inclusion for the damage function. The main aim of this paper is to show existence of weak solutions for the introduced model, where, in contrast to existing damage models in the literature, different elastic properties of damaged and undamaged material are regarded. To prove existence of weak solutions for the introduced model, we start with an approximation system. Then, by passing to the limit, existence results of weak solutions for the proposed model are obtained via suitable variational techniques.

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Keywords: Cahn–Hilliard system; Phase separation; Elliptic–parabolic systems; Doubly nonlinear differential inclusions; Complete damage; Existence results

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1. Introduction

The ongoing miniaturization in the area of micro-electronics leads to higher demands on strength and lifetime of the materials, while the structural size is continuously being reduced. Materials, which enable the functionality of technical products, change the microstructure over time. Phase separation, coarsening phenomena and damage processes take place. The complete failure of electronic devices like motherboards or mobile phones often results from micro-cracks in solder joints. Therefore, the knowledge of the mechanisms inducing phase separation, coarsening and damage phenomena is of great importance for technological applications. A uniform distribution of the original materials is aimed to guarantee evenly distributed material properties of the sample. For instance, mechanical properties, such as the strength and the stability of the material, depend on how finely regions of the original materials are mixed. The control of the evolution of the microstructure and, therefore, of the lifetime of materials relies on the ability to understand phase separation, coarsening and damage processes. Hence, a major aim is to develop reliable mathematical models for describing such effects.

Phase separation and coarsening phenomena are usually described by phase-field models of Cahn–Hilliard type. The evolution is modeled by a parabolic diffusion equation for the phase fractions. To include elastic effects, resulting from stresses caused by different elastic properties of the phases, Cahn–Hilliard systems are coupled with an elliptic equation, describing quasi-static balance of forces. Such coupled Cahn–Hilliard systems with elasticity are also called Cahn–Larché systems. Since in general the mobility, stiffness and surface tension coefficients depend on the phases (see for instance [5] and [4] for the explicit structure deduced by the embedded atom method), the mathematical analysis of the coupled problem is very complex. Existence results were derived for special cases in [14,23,25,24,3] (constant mobility, stiffness and surface tension coefficients), in [7] (concentration dependent mobility, two space dimensions), [51,52] (concentration dependent surface tension and nonlinear diffusion) and in [48] in an abstract measure-valued setting (concentration dependent mobility and surface tension tensors). For numerical results and simulations we refer e.g. to [55,37,2].

From a microscopic point of view, damage behavior originates from breaking atomic links in the material whereas a macroscopic theory may specify damage in the isotropic case by a scalar variable related to the proportion of damaged bonds in the micro-structure of the material with respect to the undamaged ones. According to the latter perspective, phase-field models are quite common to model smooth transitions between damaged and undamaged material states. Such phase-field models have been mainly investigated for incomplete damage which means that damaged material cannot loose all its elastic energy.

A first local in time existence result for a 3D damage model has been introduced in [9], where irreversibility of the damage evolution is accounted for. Damage for viscoelastic materials, in which also viscosity degenerates during the damage process, is investigated in [10]. Damage models are also analytically investigated in [45,34] and, there, existence, uniqueness and regularity properties are shown. These models do not account for temperature effect. A local in time existence result for a complete dissipative damage model with the evolving of temperature can be found in [6]. A *coupled* system describing incomplete *damage*, linear elasticity and *phase separation* appeared firstly in [28,31]. There, existence of weak solutions has been proven under mild assumptions, where, for instance, the stiffness tensor may be material-dependent and the chemical free energy may be of polynomial or logarithmic type. All these works are based on the gradient-of-damage model proposed by Frémond and Nedjar [22] (see also [21]) which describes damage as a result from microscopic movements in the solid. The distinction between a