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Averaging theory for discontinuous piecewise differential systems

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Abstract

We develop the averaging theory of first and second order for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems with the minimal conditions of differentiability. We also provide two applications.

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1. Introduction and statement of the main results

1.1. Introduction

In these last years a big interest has appeared for studying discontinuous differential systems, that is differential equations with discontinuous right-hand sides. This interest has been

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stimulated by discontinuous phenomena in control systems [2], impact and friction mechanics [7], nonlinear oscillations [1,20], economics [12,13], and biology [3,15], and it has become certainly one of the common frontiers between Mathematics, Physics and Engineering. For more details see Teixeira [22]. A recent review appears in [24].

One of the main problems in the qualitative theory of differential systems is the study of their periodic solutions. A good tool to study the periodic solutions is the averaging theory, see for instance the books of Sanders, Verhulst, and Murdock [21] and Verhulst [23]. We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behavior of nonlinear smooth dynamical systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [10]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [6] in the 1930s and Bogoliubov [5] in 1945.

The classical results in the averaging theory require, at least, that the systems are of class $\mathcal{C}^2$. Nevertheless, Buica and Llibre in [9], using mainly topological tools as the Brouwer degree theory, extended the averaging theory up to order 3 for studying periodic orbits of continuous Lipschitz differential systems. Their results were generalized for any order in [16,17]. Recently, the theory of regularization was used, by Llibre, Novaes and Teixeira in [18], to develope the averaging theory of first order for studying periodic orbits of discontinuous piecewise differential systems with two systems (pieces).

Here we develop the averaging theory of first and second order for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems (pieces). We generalize the results established in [9,18] considering minimal conditions of differentiability. Furthermore, we use this theory to study the planar linear centers perturbed by discontinuous piecewise differential systems when the set of discontinuity is composed by rays passing through the origin and when the set of discontinuity is a parabola.

1.2. Preliminaries

In what follows we define the necessary elements for the statements of our main results.

Let D be an open subset of $\mathbb{R}^d$ and $\mathbb{S}^1 = \mathbb{R}/T$ for some period $T > 0$. We consider a finite set of ODE's

$$x'(t) = f^n(t,x), \quad (t,x) \in I \times D \quad \text{for } n = 1, 2, \ldots, M, \tag{1}$$

where $f^n : \mathbb{S}^1 \times D \to \mathbb{R}^d$ is a continuous function. Here the prime denotes derivative with respect to the time t. Let (S_n) be a finite sequence of open disjoints subset of $\mathbb{S}^1 \times D$ for $n = 1, 2, \ldots, M$. We suppose that the boundaries of each S_n are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$. Furthermore the union of all boundaries, denoted by Σ, and all S_n together cover $\mathbb{S}^1 \times D$. So we define an *M-Discontinuous Piecewise Differential System* (*M-DPDS*) as

$$x'(t) = f(t,x) = \begin{cases} f^1(t,x), & (t,x) \in \overline{S}_1, \\ f^2(t,x), & (t,x) \in \overline{S}_2, \\ \quad\vdots & \\ f^M(t,x), & (t,x) \in \overline{S}_M, \end{cases} \tag{2}$$