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A sharp inequality of Trudinger–Moser type and extremal functions in $H^{1,n}(\mathbb{R}^n)$ ☆

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Abstract

We prove a sharp form of the Trudinger–Moser inequality for the Sobolev space $H^{1,n}(\mathbb{R}^n)$. The sharpness comes from the introduction of an extra factor $\|u\|_n^n$ in the classical Trudinger–Moser inequality. Let

$$\ell(\alpha) := \sup_{\{u \in H^{1,n}(\mathbb{R}^n): \|u\|_{1,n}=1\}} \int_{\mathbb{R}^n} \Phi \circ v_\alpha(u)\, \mathrm{d}x,$$

where $\Phi(t) := e^t - \sum_{i=0}^{n-1} \frac{t^i}{i!}$ and $v_\alpha(u) := \beta_n(1+\alpha\|u\|_n^n)^{1/(n-1)}|u|^{n/(n-1)}$. The main results read: (1) for $0 \le \alpha < 1$ we have $\ell(\alpha) < \infty$, (2) for $\alpha > 1$, $\ell(\alpha) = \infty$ and (3) moreover, we prove that for $0 \le \alpha < 1$, an extremal function for $\ell(\alpha)$ exists.

MSC: 35J60; 35J92; 46E35; 46E30

Keywords: Limiting Sobolev inequalities; Trudinger–Moser inequality; Sharp constants; Extremal function; Blow-up analysis

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1. Introduction

This paper is concerned with the problem of finding optimal Trudinger–Moser type inequality for the Sobolev space $H^{1,n}(\mathbb{R}^n)$ and the existence of extremal functions. Sharp Trudinger–Moser inequality plays an important role in geometric analysis and partial differential equations and continues to be a source of inspiration to many researches in recent years. In order to motivate our work, let us present a brief history of the main results on this class of problems. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a smooth bounded domain. The classical Sobolev theorem states that the imbedding $H_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q : 1 \leq q < \infty$, but $H_0^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega)$ as one can see by taking $u(r) = \ln(\ln(4R/r))$ for some $R > 0$ small enough such that $\overline{B_{2R}(0)} \subset \Omega$, where (without loss of generality) we assume $0 \in \Omega$ (cf. [3]). In this limiting case the optimal imbedding is an Orlicz space imbedding, which was proved by V.I. Yudovich [37], S.I. Pohozhaev [28], J. Peetre [27], N.S. Trudinger [35] and J. Moser [25]. More precisely, when Ω is a *bounded domain*, using the Dirichlet norm $\|\nabla u\|_n$ (equivalent to the Sobolev norm in $H_0^{1,n}(\Omega)$) they proved that

$$\sup_{\{u\in H_0^{1,n}(\Omega):\|\nabla u\|_n=1\}} \int_\Omega e^{\beta|u|^{n/(n-1)}}\, \mathrm{d}x \leq C_n|\Omega|, \tag{1.1}$$

for any $\beta \leq \beta_n := n\omega_{n-1}^{1/(n-1)}$, where $|\Omega|$ denotes the Lebesgue measure of a set Ω in $\mathbb{R}^n$ and ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Moreover, β_n is the best constant in the following sense: the integral on the left actually is finite for any positive β, but if $\beta > \beta_n$ it can be made arbitrarily large by an appropriate choice of u and the supremum is $+\infty$.

However, P.-L. Lions [23] proved that an inequality like (1.1) holds along certain sequences with a constant larger than β_n; more precisely, if $(u_k) \subset H_0^{1,n}(\Omega)$, $\|\nabla u_k\|_n = 1$ and $u_k \rightharpoonup u \not\equiv 0$ in $H_0^{1,n}(\Omega)$. Then

$$\sup_k \int_\Omega e^{p|u_k|^{n/(n-1)}}\, \mathrm{d}x < \infty \tag{1.2}$$

provided that

$$p < \frac{\beta_n}{(1 - \|\nabla u\|_n^n)^{1/(n-1)}}.$$

To complete this analysis, the following results were proposed by Adimurthi and O. Druet [4] for the case $n = 2$ and by Y. Yang [36] for the case $n \geq 3$:

$$\sup_{\{u\in H_0^{1,n}(\Omega):\|\nabla u\|_n=1\}} \int_\Omega e^{\beta_n(1+\alpha\|u\|_n^n)^{1/(n-1)}|u|^{n/(n-1)}}\, \mathrm{d}x \begin{cases} < \infty & \text{if } 0 \leq \alpha < \lambda_1(\Omega) \\ = \infty & \text{if } \alpha \geq \lambda_1(\Omega) \end{cases}$$

where $\lambda_1(\Omega) = \inf\{\|\nabla u\|_n^n : u \in H_0^{1,n}(\Omega) \text{ and } \|u\|_n = 1\}$.

The Trudinger–Moser inequalities for unbounded domains were proposed by D.M. Cao [7] for the case $n = 2$ and J.M. do Ó [14] and R. Panda [26] for general case $n \geq 2$. Precisely, if