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A localization theorem and boundary regularity for a class of degenerate Monge–Ampere equations

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Abstract

We consider degenerate Monge–Ampere equations of the type

$$\det D^2u = f \quad \text{in } \Omega, \qquad f \sim d_{\partial\Omega}^{\alpha} \quad \text{near } \partial\Omega,$$

where $d_{\partial\Omega}$ represents the distance to the boundary of the domain Ω and $\alpha > 0$ is a positive power. We obtain C^2 estimates at the boundary under natural conditions on the boundary data and the right-hand side. Similar estimates in two dimensions were obtained by J.X. Hong, G. Huang and W. Wang in [3].

1. Introduction

In this paper we discuss boundary regularity for solutions to degenerate Monge–Ampere equations of the type

$$\det D^2u = f \quad \text{in } \Omega, \qquad f \sim d_{\partial\Omega}^{\alpha} \quad \text{near } \partial\Omega,$$

where $d_{\partial\Omega}$ represents the distance to the boundary of a convex domain Ω and $\alpha > 0$ is a positive power.

Boundary estimates for the Monge–Ampere equation in the nondegenerate case $f \in C(\overline{\Omega})$, $f > 0$, were obtained starting with the works of Ivockina [4], Krylov [5], Caffarelli, Nirenberg

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and Spruck [2] (see also [1,9,10]). The general strategy for the C^2 estimates in the nondegenerate case is to obtain first a bound by above for the second derivatives on $\partial\Omega$, and then to use the equation and bound all the pure second derivatives by below. When $f = 0$ on $\partial\Omega$ this bound cannot hold since some second derivative becomes 0. In this paper we show that, under general conditions on the data, in a neighborhood of $\partial\Omega$ only one second derivative tends to 0 and all tangential pure second derivatives are continuous and bounded by below away from 0. The difficulty in proving this result lies in the fact that the tangential pure second derivatives are only subsolutions for the linearized operator, and therefore it is not clear whether or not such a lower bound is satisfied. In the case of two dimensions J.X. Hong, G. Huang and W. Wang in [3] used that the tangential second derivative is in fact a solution to an elliptic equation and showed that $u \in C^2$ up to the boundary.

In this paper we study the geometry of boundary sections in the degenerate case when f behaves in a neighborhood of $\partial\Omega$ as a positive power of the distance to $\partial\Omega$. We use the compactness methods developed in [7] where a localization theorem for boundary sections of solutions to the Monge–Ampere equation was obtained. In Theorem 2.1 we show that a localization theorem holds also in the degenerate case, and it states that boundary sections have the shape of half-ellipsoids. We achieve this by reducing the problem to the study of tangent cones for solutions to degenerate Monge–Ampere equations that have a singularity on $\partial\Omega$. Then we use the ideas from [8] where the regularity of such tangent cones was investigated for the classical Monge–Ampere equation.

Before we state our main results we recall the notion for a function to be C^2 at a point. We say that u is C^2 at x_0 if there exists a quadratic polynomial Q_{x_0} such that, in the domain of definition of u,

$$u(x) = Q_{x_0}(x) + o\big(|x - x_0|^2\big).$$

Throughout this paper we refer to a linear map A of the form

$$Ax = x + \tau x_n, \quad \text{with } \tau \cdot e_n = 0,$$

as a *sliding along* $x_n = 0$. Notice that the map A is the identity map when is restricted to $x_n = 0$ and it becomes a translation of vector $s\tau$ when is restricted to $x_n = s$.

Let Ω be a bounded convex domain such that $\partial\Omega$ is $C^{1,1}$ at the origin, that is $0 \in \partial\Omega$ and

$$\Omega \subset \{x_n > 0\}, \quad \text{and } \Omega \text{ has an interior tangent ball at the origin.} \tag{1.1}$$

We are interested in the behavior near the origin of a convex solution $u \in C(\overline{\Omega})$ to the equation

$$\det D^2u = g(x) d^{\alpha}_{\partial\Omega}, \quad \alpha > 0, \tag{1.2}$$

where g is a nonnegative function that is continuous at the origin, $g(0) > 0$.

Our main theorem is the following pointwise C^2 estimate at the boundary (see also Theorem 2.4 for a more precise quantitative version).

Theorem 1.1. *Let Ω, u satisfy (1.1), (1.2) above. Assume that*

$$u(0) = 0, \qquad \nabla u(0) = 0, \qquad u = \varphi \quad \text{on } \partial\Omega,$$

and the boundary data φ is C^2 at 0, and it separates quadratically away from 0.