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Remarks on a weighted energy estimate and its application to nonlinear wave equations in one space dimension

Makoto Nakamura

Faculty of Science, Kojirakawa-machi 1-4-12, Yamagata 990-8560, Japan

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Abstract

A weighted energy estimate with tangential derivatives on the light cone is applied for the Cauchy problem of semilinear wave equations with the null conditions in one space dimension. The well-posedness and lifespan of the solutions are considered based on the vector field method.

MSC: 35L70

Keywords: Nonlinear wave equations; Null conditions; Weighted energy estimates

1. Introduction

Let $c > 0$, $T > 0$. Let us consider the Cauchy problem of linear wave equations

$$\begin{cases} (\partial_t^2 - c^2\partial_x^2)u(t,x) = f(t,x) & \text{for } (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,\cdot) = u_0(\cdot), \quad \partial_t u(0,\cdot) = u_1(\cdot), \end{cases} \tag{1.1}$$

where u is the unknown function, f is the inhomogeneous term, u_0 and u_1 are initial data. The standard energy estimates show the inequality

E-mail address: nakamura@sci.kj.yamagata-u.ac.jp.

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$$\sup_{0\leqslant t\leqslant T}\frac{1}{2}\int_{\mathbb{R}}\bigl(\partial_t u(t,x)\bigr)^2+c^2\bigl(\partial_x u(t,x)\bigr)^2\,dx$$

$$\leqslant\frac{1}{2}\int_{\mathbb{R}}\bigl(u_1(x)\bigr)^2+c^2\bigl(\partial_x u_0(x)\bigr)^2\,dx+\int_0^T\int_{\mathbb{R}}\bigl|\partial_t u(t,x)\cdot f(t,x)\bigr|\,dx\,dt=:E, \tag{1.2}$$

where we have put the right hand side as E. We put $r:=|x|$, $\partial_r:=(x/|x|)\partial_x$, $D_c:=\partial_t+c\partial_r$ which denotes the tangential derivative on the light cone $\{(t,x)\in(0,\infty)\times\mathbb{R}\colon\ ct=|x|\}$. For $\kappa\geqslant-1$, we define a weight function $W(cT,\kappa)$ by

$$W(cT,\kappa):=\begin{cases}\{1-(1+cT)^{-\kappa}\}/\kappa & \text{if } \kappa>0 \text{ or } -1\leqslant\kappa<0,\\ \log(1+cT) & \text{if } \kappa=0.\end{cases} \tag{1.3}$$

We prepare the following weighted energy estimate.

Lemma 1.1. *The solution of (1.1) satisfies the estimate*

$$\sup_{\kappa\in\mathbb{R}}\frac{c}{12W(cT,\kappa)}\int_0^T\int_{\mathbb{R}}\frac{(D_c u(t,x))^2}{(1+|ct-r|)^{1+\kappa}}\,dx\,dt\leqslant E. \tag{1.4}$$

The weighted energy estimate (1.4) has been shown by Lindblad and Rodnianski [24, p. 76, Corollary 8.2] and Alinhac [2, Theorem 1] for three space dimensions (i.e. $x\in\mathbb{R}^3$) with $\kappa>0$, and it plays an important role to control the nonlinear terms which satisfy the null conditions since it enables us to obtain the decay estimates for waves near the light cone where the singularity propagates.

Remark 1.2. When we consider the application of Lemma 1.1, the simple bounds $W(cT,\kappa)\leqslant 1/\kappa$ if $\kappa>0$, $W(cT,\kappa)\leqslant(1+cT)^{|\kappa|}/|\kappa|$ if $-1\leqslant\kappa<0$ are useful (see the proof of Theorem 1.3, below).

Let us consider the Cauchy problem of nonlinear wave equations as the application of Lemma 1.1.

$$\begin{cases}\bigl(\partial_t^2-c^2\partial_x^2\bigr)u(t,x)=f(\partial_t u,\partial_x u)(t,x) & \text{for } (t,x)\in[0,T)\times\mathbb{R},\\ u(0,\cdot)=u_0(\cdot),\quad \partial_t u(0,\cdot)=u_1(\cdot),\end{cases} \tag{1.5}$$

where $f(\partial_t u,\partial_x u)$ denotes the nonlinear term dependent on $\partial_t u$ and $\partial_x u$. We use the vector fields

$$\partial_t,\quad \partial_x,\quad \Omega_c:=ct\partial_x+\frac{x}{c}\partial_t,\quad L:=t\partial_t+r\partial_r,\quad \Gamma_c:=(\partial_t,\partial_x,\Omega_c,L), \tag{1.6}$$

where $r:=|x|$. We put $\Box_c:=\partial_t^2-c^2\partial_x^2$ and note the commuting properties $\partial_t\Omega_c=\Omega_c\partial_t+c\partial_x$, $\partial_x\Omega_c=\Omega_c\partial_x+\partial_t/c$, $\partial_t L=(L+1)\partial_t$, $\partial_x L=(L+1)\partial_x$, $L\Omega_c=\Omega_c L$, $\Box_c\partial_{t,x}=\partial_{t,x}\Box_c$, $\Box_c\Omega_c=\Omega_c\Box_c$, $\Box_c L=(L+2)\Box_c$. We denote the size of initial data by