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J. Differential Equations 256 (2014) 407–460

Journal of Differential Equations

www.elsevier.com/locate/jde

The existence and singularity structures of low regularity solutions to higher order degenerate hyperbolic equations

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Received 4 May 2013

Available online 9 October 2013

Abstract

This paper is a continuation of our partial work in [21], where we established that the bounded local solution $u(t, x)$ exists and is piecewise smooth for the second order degenerate hyperbolic equation $(\partial_t^2 - t^m \Delta_x)u = f(t, x, u)$ with the initial data of C^1-piecewise smooth $u(0, x)$ and piecewise smooth $\partial_t u(0, x)$. In the present paper, we will consider the lower regularity solution of the higher order degenerate hyperbolic equation in the category of discontinuous and even unbounded functions.

MSC: 35L70; 35L65; 35L67; 76N15

Keywords: Higher order degenerate hyperbolic equation; Hilbert transform; Piecewise smooth; Confluent hypergeometric function; Cusp singularity; Conormal space

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[1] Ruan Zhuoping and Yin Huicheng are supported by the NSFC (No. 1093-1007, No. 11025105), by the Priority Academic Program Development of Jiangsu Higher Education Institutions, and by the DFG via the Sino-German project "Analysis of PDEs and application".

[2] Ingo Witt was partly supported by the DFG via the Sino-German project "Analysis of PDEs and application.

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.09.007

1. Introduction

In this paper, we will study the local existence and singularity structures of low regularity solutions to the following higher order degenerate hyperbolic equations

$$\begin{cases} \partial_t\big(\partial_t^2 - t^m \Delta_x\big)u = f(t,x,u), & (t,x) \in [0,+\infty) \times \mathbb{R}^n, \\ \partial_t^j u(0,x) = \varphi_j(x), & 0 \leqslant j \leqslant 2 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \big(\partial_t^2 - t^{m_1}\Delta_x\big)\big(\partial_t^2 - t^{m_2}\Delta_x\big)u = f(t,x,u), & (t,x) \in [0,+\infty) \times \mathbb{R}^n, \\ \partial_t^k u(0,x) = \psi_k(x), & 0 \leqslant k \leqslant 3, \end{cases} \tag{1.2}$$

where $m, m_1, m_2 \in \mathbb{N}$, $m_1 \neq m_2$, $x \in \mathbb{R}^n$, f is C^∞ smooth on its arguments and has a compact support on the variable $x = (x_1, \ldots, x_n)$, and the discontinuous initial data $\varphi_j(x)$ $(0 \leqslant j \leqslant 2)$ and $\psi_k(x)$ $(0 \leqslant k \leqslant 3)$ satisfy one of the assumptions:

(A_1) $$\varphi_j(x) = \begin{cases} \varphi_{j1}(x) & \text{for } x_1 > 0, \\ \varphi_{j2}(x) & \text{for } x_1 < 0, \end{cases} \qquad \psi_k(x) = \begin{cases} \psi_{k1}(x) & \text{for } x_1 > 0, \\ \psi_{k2}(x) & \text{for } x_1 < 0, \end{cases}$$

here $\varphi_{j1}, \varphi_{j2}, \psi_{k1}, \psi_{k2} \in C_0^\infty(\mathbb{R}^n)$ with $\varphi_{j1}(0) \neq \varphi_{j2}(0)$ and $\psi_{k1}(0) \neq \psi_{k2}(0)$.

(A_2) $\varphi_j(x) = g_j(x, \frac{x}{|x|})$, $\psi_k(x) = h_k(x, \frac{x}{|x|})$, here $g_j(x,y)$ and $h_k(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ have compact supports in $B(0,1) \times B(0,2)$.

Under the assumptions (A_1) and (A_2), we could get the following main results.

Theorem 1.1. *Under the condition (A_1), there exists a constant $T > 0$ such that*

(i) (1.1) *has a unique solution $u \in L^\infty([0,T] \times \mathbb{R}^n) \cap C([0,T], H^{\frac{1}{2}-}(\mathbb{R}^n)) \cap C((0,T], H^{\frac{m+1}{m+2}-}(\mathbb{R}^n)) \cap C^1([0,T], H^{-\frac{1}{m+2}-}(\mathbb{R}^n))$ and $u \in C^\infty((0,T] \times \mathbb{R}^n \setminus \Gamma_m^\pm \cup \Gamma_0)$, where $\Gamma_m^\pm = \{(t,x)\colon t \geqslant 0, x_1 = \pm\frac{2t^{(m+2)/2}}{m+2}\}$, and $\Gamma_0 = \{(t,x)\colon t \geqslant 0, x_1 = 0\}$. In addition, here and below $H^{s-} = \bigcap_{\varepsilon>0} H^{s-\varepsilon}$.*

(ii) (1.2) *has a unique solution $u \in L^\infty([0,T] \times \mathbb{R}^n) \cap C([0,T], H^{\frac{1}{2}-}(\mathbb{R}^n)) \cap C((0,T], H^{\frac{m_2+1}{m_2+2}-}(\mathbb{R}^n)) \cap C^1([0,T], H^{-\frac{1}{m_2+2}-}(\mathbb{R}^n))$ and $u \in C^\infty((0,T] \times \mathbb{R}^n \setminus \Gamma_{m_1}^\pm \cup \Gamma_{m_2}^\pm)$, where $\Gamma_{m_i}^\pm = \{(t,x)\colon t \geqslant 0, x_1 = \pm\frac{2t^{(m_i+2)/2}}{m_i+2}\}$ for $i = 1, 2$.*

Theorem 1.2. *Under the condition (A_2), and further suppose that f satisfies $|\partial_{t,x}^\alpha \partial_u^l f(t,x,u)| \leqslant C_{T_0,\alpha,l}(1+|u|)^{\max\{K-l,0\}}$ for $|\alpha|, l \in \mathbb{N} \cup \{0\}$ and $0 \leqslant t \leqslant T_0$, here $T_0 > 0$ and $K > 0$ are some fixed constants, then there exists a constant $T > 0$ ($T \leqslant T_0$) such that*

(i) (1.1) *has a unique solution $u \in L_{loc}^\infty((0,T] \times \mathbb{R}^n) \cap C([0,T], H^{\frac{n}{2}-}(\mathbb{R}^n)) \cap C((0,T], H^{\frac{n}{2}+\frac{m}{2(m+2)}-}(\mathbb{R}^n)) \cap C^1([0,T], H^{\frac{n}{2}-\frac{m+4}{2(m+2)}-}(\mathbb{R}^n))$, and $u \in C^\infty((0,T] \times \mathbb{R}^n \setminus \Gamma_m \cup l_0)$, where $\Gamma_m = \{(t,x)\colon t \geqslant 0, |x|^2 = \frac{4t^{m+2}}{(m+2)^2}\}$, and $l_0 = \{(t,x)\colon t \geqslant 0, |x| = 0\}$.*