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Non-integrability of measure preserving maps via Lie symmetries

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Abstract

We consider the problem of characterizing, for certain natural number m, the local $\mathcal{C}^m$-non-integrability near elliptic fixed points of smooth planar measure preserving maps. Our criterion relates this non-integrability with the existence of some Lie Symmetries associated to the maps, together with the study of the finiteness of its periodic points. One of the steps in the proof uses the regularity of the period function on the whole period annulus for non-degenerate centers, question that we believe that is interesting by itself. The obtained criterion can be applied to prove the local non-integrability of the Cohen map and of several rational maps coming from second order difference equations.

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1. Introduction and main results

In the last years the development of criteria to determine the integrable nature of discrete dynamical systems has been the focus of an intensive research activity (see [14] and references therein), however there are very few non-integrability results for discrete dynamical systems, see for instance [8,5,10,12,25,27] and their references. The main result of this paper, Theorem 1 below, provides a criterion to establish the local non-integrability of real planar measure preserving maps in terms of non-existence of local first integrals of class $\mathcal{C}^m$, for certain $m \in \mathbb{N}$, near an *elliptic* fixed point (that is, a fixed point such that the eigenvalues of the associated linear part lie in the unit circle, but excluding the values ± 1).

We will say that a planar map is *$\mathcal{C}^m$-locally integrable at an elliptic fixed point p* if there do exist a neighborhood $\mathcal{U}$ of p and a locally non-constant real-valued function $V \in \mathcal{C}^m(\mathcal{U})$, with $m \geq 2$ (called first integral) such that $V(F(x)) = V(x)$, all the level curves $\{V = h\} \cap \mathcal{U}$ are closed curves surrounding p and, moreover, p is an isolated non-degenerate critical point of V in $\mathcal{U}$.

Prior to state the main result, we recall that a map F defined on $\mathcal{U}$, an open set of $\mathbb{R}^2$, preserves an absolutely continuous measure with respect to the Lebesgue's one with non-vanishing density ν, if $m(F^{-1}(B)) = m(B)$ for any measurable set B, where $m(B) = \int_B \nu(x, y)\, dx dy$, and $\nu|_{\mathcal{U}} \neq 0$. For the sake of simplicity, in this paper sometimes we will refer these maps simply as *measure preserving maps*.

When the eigenvalues $\lambda, \bar{\lambda} = 1/\lambda$ of the linear part of a $\mathcal{C}^1$-planar map F at an elliptic fixed point $p \in \mathbb{R}^2$ are not roots of unity of order ℓ for $0 < \ell \leq k$ we will say that p is *not k-resonant*. Recall that a $\mathcal{C}^{k+1}$-map, F, with not k-resonant elliptic fixed points, is locally conjugated to its *Birkhoff normal form* plus some remainder terms, see [1]:

$$F_B(z) = \lambda z \left(1 + \sum_{j=1}^{[(k-1)/2]} B_j (z\bar{z})^j\right) + O(|z|^{k+1}), \tag{1}$$

where $z = x + iy$, and $[\cdot]$ denotes the integer part. It is well-known that near a locally integrable elliptic point the first non-vanishing *Birkhoff constant* $B_n \in \mathbb{C}$, if exists, must be purely imaginary. We recall a proof of this fact in Lemma 12.

The main result of this paper is the following theorem:

Theorem 1. *Let F be a $\mathcal{C}^{2n+2}$-planar map defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$ with an elliptic fixed point p, not $(2n+1)$-resonant, and such that its first non-vanishing Birkhoff constant is $B_n = i\, b_n$, for some $0 < n \in \mathbb{N}$ and $b_n \in \mathbb{R} \setminus \{0\}$. Moreover, assume that F is a measure preserving map with a non-vanishing density $\nu \in \mathcal{C}^{2n+3}$. If, for an unbounded sequence of natural numbers $\{N_k\}_k$, F has finitely many N_k-periodic points in $\mathcal{U}$ then it is not $\mathcal{C}^{2n+4}$-locally integrable at p.*

Our proof uses some of the ideas presented by G. Lowther in [20] for explaining the non-integrability of the Cohen map. As we will see, our result has also several applications for proving non-smooth integrability of several rational difference equations.

One of the main ingredients in our proof of Theorem 1 is that any integrable measure preserving map has an associated vector field X, called a *Lie Symmetry*, such that F can be expressed in terms of the flow of X, see Section 3 for further details. As we will see, to proceed with our approach, from this Lie symmetry we need to construct another one, say Y, having an isochronous