Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 5406–5439

Journal of Differential Equations

www.elsevier.com/locate/jde

Fast solutions and asymptotic behavior in a reaction–diffusion equation ☆

Margarita Arias, Juan Campos *

Universidad de Granada, Departamento Matemática Aplicada, Facultad de Ciencias, Avd Fuentenueva s/n, 18071, Granada, Spain

Received 10 January 2014; revised 18 June 2015

Available online 13 July 2015

Abstract

In this paper we deal with the existence of traveling waves solutions (t.w.s.) for the reaction–diffusion equation

$$u_t = u_{xx} + f(u),$$

in a very general setting of reaction terms f with two distinguished stationary states, say 0 and 1. We link the existence of some type of solutions of the second order ODE

$$u'' + cu' + f(u) = 0,$$

with the existence of fast t.w.s. By defining fast solutions for this ODE, we find a value $c_M > 0$ related to the existence of global fast solutions and determine c_M through a variational formula. Our results allow us particularly to show that any solution $u(x,t)$ of the reaction–diffusion equation with compactly supported initial data and $0 \le u(x,0) \le 1,\ x \in \mathbb{R}$, satisfies

$$\lim_{t \to +\infty} u(x+ct,t) = 0,$$

uniformly on compact sets, for all c, $|c| > c_M$.

Finally, we connect c_M with the minimum speed of propagation of all t.w.s. c^*.

☆ The authors are partially supported by MINECO (Spain), project MTM2011-23652 and project MTM2014-53406-R.
* Corresponding author.
E-mail addresses: marias@ugr.es (M. Arias), campos@ugr.es (J. Campos).

http://dx.doi.org/10.1016/j.jde.2015.06.036
0022-0396/

MSC: 35K57; 35A15; 35C07

Keywords: Reaction–diffusion equations; Traveling wave solutions; Wave speed; Fast heteroclinic

1. Introduction

Many phenomena in Biology, Chemistry or Physics dealing with dissipative dynamical systems are modeled by reaction–diffusion equations that in one-dimensional space may look like

$$u_t = u_{xx} + f(u), \tag{1}$$

where the function $f(u)$ represents the kinetics. For many applications $f(u)$ is a nonlinear function which vanishes at two distinguished points, say 0 and 1, and only solutions between these two stationary states are interesting.

One of the simplest cases is when $f(u) = au(1-u)$, $a > 0$. This reaction term was used by Fisher [9] in the 1930's to study the propagation of a dominant gene in a population. In this case, $u(x,t)$ represents the concentration of population with the chosen gene in the point x at the instant t, so that only solutions of (1) between the two stationary states $u = 0$ and $u = 1$ are interesting.

When there is no diffusion and $u(x,t)$ is a function of t, equation (1) becomes the first order ordinary equation

$$u' = f(u). \tag{2}$$

It is well known that any solution of (2) with

$$0 \le u \le 1 \tag{3}$$

is defined on all the real line and has a simple dynamic:

$$\lim_{t\to\pm\infty} u(t) = \alpha_\pm,$$

where $\alpha_\pm$ are stationary states of the equation, that is, $f(\alpha_\pm) = 0$.

The appearance of diffusion entails a greater complexity on the dynamic of the equation. Nowadays it is well known that the asymptotical behavior of the solutions of (1) is closely related to the existence of *traveling waves (or fronts)* connecting two stationary states. By a traveling wave solution (t.w.s.) we mean a solution of (1) and (3) of the form $u(x,t) = u(x - ct)$, for some constant c.

The function $u(\zeta)$ is called the *profile of the wave* and c is also an unknown constant which is usually referred to as the *speed of propagation.*

Fisher, in [9], showed that equation (1) with $f(u) = au(1-u)$ has t.w.s. moving with speed c for all $c \ge c^* = 2\sqrt{a}$. In their classical paper [13], Kolmogorov, Petrovsky and Piscounoff considered reaction terms $f \in C^1$ satisfying $f'(0) > 0$, $f'(u) \le f'(0)$, $f(u) > 0$, $u \in (0,1)$. They proved the existence of a number $c^* = 2\sqrt{f'(0)}$ so that equation (1) possesses t.w.s. moving with speed c for all c, $|c| \ge c^*$. Moreover, they also proved that the solution of (1) satisfying