Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 5440–5485

Journal of Differential Equations

www.elsevier.com/locate/jde

Global small solutions of 2-D incompressible MHD system

Fanghua Lin [a], Li Xu [b], Ping Zhang [c,*]

[a] Courant Institute, New York University, New York, NY 10012, United States

[b] LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, CAS, Beijing 100190, China

[c] Academy of Mathematics and Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China

Received 27 September 2014

Available online 3 July 2015

Abstract

In this paper, we consider the global wellposedness of 2-D incompressible magneto-hydrodynamical system with smooth initial data which is close to some non-trivial steady state. It is a coupled system between the Navier–Stokes equations and a free transport equation with a universal nonlinear coupling structure. The main difficulty of the proof lies in exploring the dissipative mechanism of the system. To achieve this and to avoid the difficulty of propagating anisotropic regularity for the free transport equation, we first reformulate our system (1.1) in the Lagrangian coordinates (2.19). Then we employ anisotropic Littlewood–Paley analysis to establish the key *a priori* $L^1(\mathbb{R}^+; Lip(\mathbb{R}^2))$ estimate for the Lagrangian velocity field Y_t. With this estimate, we can prove the global wellposedness of (2.19) with smooth and small initial data by using the energy method. We emphasize that the algebraic structure of (2.19) is crucial for the proofs to work. The global wellposedness of the original system (1.1) then follows by a suitable change of variables.

MSC: 35Q30; 76D03

Keywords: Inviscid MHD system; Anisotropic Littlewood–Paley theory; Dissipative estimates; Lagrangian coordinates

* Corresponding author.
E-mail addresses: linf@cims.nyu.edu (F. Lin), xuliice@lsec.cc.ac.cn (L. Xu), zp@amss.ac.cn (P. Zhang).

http://dx.doi.org/10.1016/j.jde.2015.06.034

1. Introduction

In this paper, we investigate the global wellposedness of the following 2-D incompressible magneto-hydrodynamical system:

$$\begin{cases} \partial_t \phi + \boldsymbol{u}\cdot\nabla\phi = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = -\mathrm{div}\big[\nabla\phi\otimes\nabla\phi\big], \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \phi|_{t=0} = \phi_0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \tag{1.1}$$

with initial data $(\phi_0, \boldsymbol{u}_0)$ smooth and close enough to the equilibrium state $(x_2, \mathbf{0})$. Here ϕ denotes the magnetic potential and $\boldsymbol{u} = (u^1, u^2)^T$, p is the velocity and scalar pressure of the fluid respectively.

Recall that the general MHD system in $\mathbb{R}^d$ reads

$$\begin{cases} \partial_t \boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \\ \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = -\frac{1}{2}\nabla|\boldsymbol{b}|^2 + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \\ \mathrm{div}\,\boldsymbol{u} = \mathrm{div}\,\boldsymbol{b} = 0, \\ \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \tag{1.2}$$

where $\boldsymbol{b} = (b^1, \dots, b^d)^T$ denotes the magnetic field, and $\boldsymbol{u} = (u^1, \dots, u^d)^T$, p is the velocity and scalar pressure of the fluid respectively. This MHD system (1.2) with zero diffusivity in the equation for the magnetic field can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. It often applies to the case when one is interested in the k-length scales that are much longer than the ion skin depth and the Larmor radius perpendicular to the field, long enough along the field to ignore the Landau damping, and time scales much longer than the ion gyration time [14,20,4]. In the particular case when $d = 2$ in (1.2), $\mathrm{div}\,\boldsymbol{b} = 0$ implies the existence of a scalar function ϕ so that $\boldsymbol{b} = (\partial_2\phi, -\partial_1\phi)^T$, and the corresponding system becomes (1.1).

It is a long standing open problem that whether or not classical solutions of (1.2) can develop finite time singularities even in the 2-D case. Except with full magnetic diffusion in (1.2), the corresponding 2-D system possesses a unique global smooth solution (see [16,31,1] for initial data in the critical spaces). With mixed partial dissipation and additional (artificial) magnetic diffusion in the 2-D MHD system, Cao and Wu [5] (see also [6]) proved its global wellposedness for any data in $H^2(\mathbb{R}^2)$. In [25], we proved the global wellposedness of a three dimensional version of (1.1) with smooth initial data which is close to a non-trivial steady state. The aim of this paper is to establish the global existence and uniqueness of solutions to the MHD equation (1.1) in the 2-D case with the same class of the initial data.

We note that the system (1.1) has appeared in many problems, see the recent survey article [22]. For the inviscid, incompressible MHD equations (1.2), it is an important problem that if it possesses a dissipation mechanism even though the magnetic diffusivity is close to zero. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales much larger than the dissipative ones. Indeed it has been conjectured that in the MHD systems, energy is dissipated at a rate that is independent of the ohmic resistivity [10].