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Harnack inequality for degenerate and singular elliptic equations with unbounded drift

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Abstract

We prove a Harnack inequality for functions which, at points of large gradient, are solutions of elliptic equations with unbounded drift.

Keywords: Harnack inequality; Nondivergence equation; Degenerate elliptic

1. Introduction

In this paper we consider operators of the form

$$Lu = a^{ij}(x)u_{ij} + b^i(x)u_i$$

on $B_1 \subset \mathbb{R}^n$, $n \geq 2$, where

$$\lambda I \leq a^{ij}(x) \leq \Lambda I$$

are uniformly elliptic, bounded measurable coefficients, and the drift $b = (b^1, ..., b^n)$ is in $L^n(B_1)$ with

$$\|b\|_{L^n(B_1)} = S.$$

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We study functions which are solutions to elliptic equations, but only at points where the gradient is large:

Definition 1.1. Assume $f \in L^n(B_1)$, and L is as above. We say $u \in W^{2,n}(B_1)$ solves

$$L_\gamma u = f$$

for some $\gamma \geq 0$ if $Lu = f$, but only where $|\nabla u| > \gamma$ (in the Lebesgue point sense).

Imbert and Silvestre recently studied the case when $|b| \in L^\infty$ in [5]. The authors prove that such functions satisfy a Harnack inequality and are Hölder continuous. The idea is that the function is already regular where the gradient is small, and where the gradient is large it solves an equation. The difficulty is that we don't know a priori where the gradient is large. The key step is an ABP-type estimate which says that if a positive solution is small at some point, then it is small in a set of positive measure. In [5] the authors obtain this estimate by sliding cusps from below the graph of u until they touch, which ensures that the equation holds at contact points, and estimating the measure of these contact points.

Our first main contribution in this paper is a new proof of the measure estimate in [5] that uses sliding of paraboloids from below at all scales and a set decomposition algorithm (see Proposition 3.1). While our technique for proving Proposition 3.1 is slightly more involved than sliding cusps, it more directly captures the dichotomy between contact points at large gradient where the equation holds, and contact points at small gradient where we can rescale to the original situation.

Savin used the idea of applying the equation at contact points with paraboloids in [7] to prove an ABP-type measure estimate. Wang subsequently adapted this to the parabolic setting in [8]. It seems hopeful that our technique can also be adapted to prove an analogous measure estimate for a class of degenerate parabolic equations. The sliding cusps technique seems difficult to extend to the parabolic setting.

In the remaining parts of this paper we extend the results of [5] to the situation of unbounded drift. Heuristically, to get estimates depending on $\|b\|_{L^{n+\epsilon}(B_1)}$ for any $\epsilon > 0$ is no different from doing the usual Krylov–Safonov theory, since under the rescaling $\tilde{u}(x) = u(rx)$ our equation becomes

$$a^{ij}(rx)\tilde{u}_{ij} + rb^i(rx)\tilde{u}_i = r^2 f,$$

so the new drift term has $L^{n+\epsilon}$ norm $r^{\epsilon/(n+\epsilon)}\|b\|_{L^{n+\epsilon}(B_r)}$, and thus doesn't come into play for r small.

On the other hand, we cannot expect to get estimates depending on $\|b\|_{L^{n-\epsilon}(B_1)}$, where rescaling makes the drift term "larger". Indeed, take the example $\frac{1}{2}|x|^2$, which solves the equation

$$\Delta u - \frac{nx}{|x|^2} \cdot \nabla u = 0.$$

In this simple example $|b| = \frac{n}{|x|}$ which is in $L^{n-\epsilon}$ for any $\epsilon > 0$ but not in L^n. In this example we violate the strong maximum principle, a qualitative version of the Harnack inequality. Thus,