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J. Differential Equations 258 (2015) 1653–1684

Journal of Differential Equations

www.elsevier.com/locate/jde

Compressible hydrodynamic flow of nematic liquid crystals with vacuum

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Received 18 January 2014; revised 25 September 2014

Available online 3 December 2014

Abstract

In this paper, we first consider the free boundary problem for a simplified version of Ericksen–Leslie equations modeling the compressible hydrodynamic flow of nematic liquid crystals in dimension one which connects continuously to vacuum. We obtain the existence of global weak solutions. Furthermore, we establish the life-span of smooth solutions to the compressible nematic liquid crystal model with the support of density growing sublinearly in time direction.

MSC: 76N10; 35Q30; 35R35

Keywords: Free boundary problem; Nematic liquid crystals; Compressible hydrodynamic flow; Weak solutions; Smooth solutions

1. Introduction

In this paper, we consider one-dimensional free boundary problem for compressible nematic liquid crystal flow which connects continuously to vacuum. The problem consists of the following equations for unknown functions $(\rho, u, n) : [a(t), b(t)] \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R} \times S^2$:

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http://dx.doi.org/10.1016/j.jde.2014.11.008
0022-0396/

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x + \left(P(\rho)\right)_x = \mu u_{xx} - \lambda\left(|n_x|^2\right)_x, \\ n_t + u n_x = \theta\left(n_{xx} + |n_x|^2 n\right) \end{cases} \tag{1.1}$$

for $(x,t) \in (a(t), b(t)) \times (0, +\infty)$ with the initial conditions:

$$(\rho, u, n)|_{t=0} = \left(\rho_0(x), u_0(x), n_0(x)\right) \quad \text{in } \left[a(0), b(0)\right], \tag{1.2}$$

where $n_0(x) : [a(0), b(0)] \to S^2$, the unit sphere in $\mathbb{R}^3$. Here, $a(t)$ and $b(t)$ are the free boundaries, i.e., the interface of the material of liquid crystal and vacuum, satisfying

$$\begin{cases} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t} = u\big(a(t), t\big), \quad t \geq 0, \qquad a(0) = a_0 \quad (a_0 > 0), \\ \dfrac{\mathrm{d}b(t)}{\mathrm{d}t} = u\big(b(t), t\big), \quad t \geq 0, \qquad b(0) = b_0 \quad (b_0 > 0). \end{cases} \tag{1.3}$$

It is interesting to study the case when ρ_0 is compactly supported and connects to vacuum continuously. In this case, the boundary conditions should be

$$\rho\big(a(t), t\big) = \rho\big(b(t), t\big) = 0 \tag{1.4}$$

for $t \geq 0$. Here ρ denotes the density, u denotes the velocity field, n denotes the macroscopic average of the nematic liquid crystal orientation field that is a unit vector (i.e., $|n| = 1$), $\mu > 0$, $\lambda > 0$, $\theta > 0$ are viscosity of the fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. $P = P(\rho)$ is the pressure. For simplicity, we assume that $P = A\rho^\gamma$ for some constants $\gamma > 1$ and $A > 0$.

The nematic liquid crystal flow is described by a system that couples a forced Navier–Stokes equations with the transported flow of harmonic maps. There have been many works interested in the free boundary problem of compressible Navier–Stokes equations. The first consideration is the case when the viscosity coefficient μ is a constant and the initial density is away from vacuum, the global existence of unique solution was established by Kazhikhov [14] and then the asymptotic behavior of the solution was studied by Okada [22] in one dimension. After that, the free boundary problem with one boundary fixed and the other connected to vacuum was investigated in [23] where the global existence of the weak solutions was proved. Similar results were obtained in [24] for the spherically symmetric motion. A further understanding of the regularity and behavior of solutions near the interfaces between the gas and vacuum was given in [20].

The second consideration is the case when the viscosity μ depends on the density ρ (i.e., $\mu(\rho) = A\rho^\alpha$). If the density function connects to vacuum with jump discontinuities, the local existence of weak solutions to Navier–Stokes equations with vacuum was studied in [19]. Jiang in [12] proved the global existence of smooth solutions provided that $0 < \alpha < \frac{1}{4}$. Okada, Matušů-Nečasová and Makino [25] obtained the existence of global weak solutions for $0 < \alpha < \frac{1}{3}$. Their result was generalized to the case $0 < \alpha < \frac{1}{2}$ by Yang, Yao and Zhu [30]. In [25,30], the initial data are required to satisfy $\rho_0, \partial_x u_0 \in \mathrm{Lip}(I)$. Then, Jiang, Xin and Zhang [13] extended the results in [25,30] to the case $0 < \alpha < 1$ with initial data satisfying $\rho_0 \in W^{1,p}(I)$ and $u_0 \in L^p(I)$ for some p. Furthermore, they proved the uniqueness provided that $u_0 \in H^1(I)$. Concerned with the dynamical behaviors, the interested readers can refer to [16] and references therein.